Exercise 5.1

Question 1:

Prove that the function \( f(x) = 5x - 3 \) is continuous at \( x = 0 \), at \( x = -3 \) and at \( x = 5 \).

Answer

The given function is \( f(x) = 5x - 3 \)

At \( x = 0 \), \( f(0) = 5 \times 0 - 3 = -3 \)

\[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} (5x - 3) = 5 \times 0 - 3 = -3 \]

\[ \therefore \lim_{{x \to 0}} f(x) = f(0) \]

Therefore, \( f \) is continuous at \( x = 0 \)

At \( x = -3 \), \( f(-3) = 5 \times (-3) - 3 = 18 \)

\[ \lim_{{x \to -3}} f(x) = \lim_{{x \to -3}} (5x - 3) = 5 \times (-3) - 3 = -18 \]

\[ \therefore \lim_{{x \to -3}} f(x) = f(-3) \]

Therefore, \( f \) is continuous at \( x = -3 \)

At \( x = 5 \), \( f(5) = 5 \times 5 - 3 = 25 - 3 = 22 \)

\[ \lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} (5x - 3) = 5 \times 5 - 3 = 22 \]

\[ \therefore \lim_{{x \to 5}} f(x) = f(5) \]

Therefore, \( f \) is continuous at \( x = 5 \)

Question 2:

Examine the continuity of the function \( f(x) = 2x^2 - 1 \) at \( x = 3 \).

Answer

The given function is \( f(x) = 2x^2 - 1 \)

At \( x = 3 \), \( f(3) = 2 \times 3^2 - 1 = 17 \)

\[ \lim_{{x \to 3}} f(x) = \lim_{{x \to 3}} (2x^2 - 1) = 2 \times 3^2 - 1 = 17 \]

\[ \therefore \lim_{{x \to 3}} f(x) = f(3) \]

Thus, \( f \) is continuous at \( x = 3 \)
Question 3:
Examine the following functions for continuity.

(a) \( f(x) = x - 5 \)
(b) \( f(x) = \frac{1}{x - 5}, x \neq 5 \)
(c) \( f(x) = \frac{x^2 - 25}{x + 5}, x \neq -5 \)
(d) \( f(x) = |x - 5| \)

Answer

(a) The given function is \( f(x) = x - 5 \)
It is evident that \( f \) is defined at every real number \( k \) and its value at \( k \) is \( k - 5 \).
It is also observed that, \( \lim_{x \to k} f(x) = \lim_{x \to k} (x - 5) = k - 5 = f(k) \)
\( \therefore \lim_{x \to k} f(x) = f(k) \)
Hence, \( f \) is continuous at every real number and therefore, it is a continuous function.

(b) The given function is \( f(x) = \frac{1}{x - 5}, x \neq 5 \)
For any real number \( k \neq 5 \), we obtain
\( \lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x - 5} = \frac{1}{k - 5} \)
Also, \( f(k) = \frac{1}{k - 5} \) (As \( k \neq 5 \))
\( \therefore \lim_{x \to k} f(x) = f(k) \)
Hence, \( f \) is continuous at every point in the domain of \( f \) and therefore, it is a continuous function.

(c) The given function is
\( f(x) = \frac{x^2 - 25}{x + 5}, x \neq -5 \)
For any real number \( c \neq -5 \), we obtain
Hence, \( f \) is continuous at every real number and therefore, it is a continuous function.

\[
f(x) = |x - 5| = \begin{cases} 
5 - x, & \text{if } x < 5 \\
5 - x, & \text{if } x \geq 5
\end{cases}
\]

(d) The given function is defined at all points of the real line.

Let \( c \) be a point on a real line. Then, \( c < 5 \) or \( c = 5 \) or \( c > 5 \)

Case I: \( c < 5 \)

Then, \( f(c) = 5 - c \)

\[
\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (5 - x) = 5 - c
\]

\[
\therefore \lim_{{x \to c}} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all real numbers less than 5.

Case II: \( c = 5 \)

Then, \( f(c) = f(5) = (5 - 5) = 0 \)

\[
\lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} (5 - x) = (5 - 5) = 0
\]

\[
\lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} (x - 5) = 0
\]

\[
\therefore \lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} f(x) = f(c)
\]

Therefore, \( f \) is continuous at \( x = 5 \)

Case III: \( c > 5 \)

Then, \( f(c) = f(5) = c - 5 \)

\[
\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (x - 5) = c - 5
\]

\[
\therefore \lim_{{x \to c}} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all real numbers greater than 5.

Hence, \( f \) is continuous at every real number and therefore, it is a continuous function.
Question 4:

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where $n$ is a positive integer.

Answer

The given function is $f(x) = x^n$

It is evident that $f$ is defined at all positive integers, $n$, and its value at $n$ is $n^n$.

Then, $\lim_{x \to n} f(n) = \lim_{x \to n}(x^n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)$

Therefore, $f$ is continuous at $n$, where $n$ is a positive integer.

Question 5:

Is the function $f$ defined by

$$f(x) = \begin{cases} 
  x, & \text{if } x \leq 1 \\
  5, & \text{if } x > 1
\end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer

The given function $f$ is

At $x = 0$,

It is evident that $f$ is defined at 0 and its value at 0 is 0.

Then, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, $f$ is continuous at $x = 0$

At $x = 1$,

$f$ is defined at 1 and its value at 1 is 1.

The left hand limit of $f$ at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

The right hand limit of $f$ at $x = 1$ is,
\[
\lim_{{x \to 1}} f(x) = \lim_{{x \to 1}} (5) = 5
\]
\[
\therefore \lim_{{x \to 1}} f(x) \neq \lim_{{x \to 1}} f(x)
\]
Therefore, \( f \) is not continuous at \( x = 1 \)

At \( x = 2 \),
\( f \) is defined at 2 and its value at 2 is 5.

Then,
\[
\lim_{{x \to 2}} f(x) = \lim_{{x \to 2}} (5) = 5
\]
\[
\therefore \lim_{{x \to 2}} f(x) = f(2)
\]
Therefore, \( f \) is continuous at \( x = 2 \)

**Question 6:**
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
2x + 3, & \text{if } x \leq 2 \\
2x - 3, & \text{if } x > 2 
\end{cases}
\]

**Answer**

The given function \( f \) is

It is evident that the given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line. Then, three cases arise.

(i) \( c < 2 \)

(ii) \( c > 2 \)

(iii) \( c = 2 \)

**Case (i) \( c < 2 \)**

Then,
\[
f(c) = 2c + 3
\]
\[
\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (2x + 3) = 2c + 3
\]
\[
\therefore \lim_{{x \to c}} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x < 2 \)

**Case (ii) \( c > 2 \)**
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 2 \)

Case (iii) \( c = 2 \)

Then, the left hand limit of \( f \) at \( x = 2 \) is,

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x + 3) = 2 \times 2 + 3 = 7
\]

The right hand limit of \( f \) at \( x = 2 \) is,

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (2x - 3) = 2 \times 2 - 3 = 1
\]

It is observed that the left and right hand limit of \( f \) at \( x = 2 \) do not coincide.
Therefore, \( f \) is not continuous at \( x = 2 \)
Hence, \( x = 2 \) is the only point of discontinuity of \( f \).

**Question 7:**
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
|x| + 3, & \text{if } x \leq -3 \\
-2x, & \text{if } -3 < x < 3 \\
6x + 2, & \text{if } x \geq 3
\end{cases}
\]

**Answer**

\[
f(x) = \begin{cases} 
|x| + 3 = -x + 3, & \text{if } x \leq -3 \\
-2x, & \text{if } -3 < x < 3 \\
6x + 2, & \text{if } x \geq 3
\end{cases}
\]

The given function \( f \) is

The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

**Case I:**
If \( c < -3 \), then \( f(c) = -c + 3 \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (-x + 3) = -c + 3
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x < -3 \)

**Case II:**

If \( c = -3 \), then \( f(-3) = -(−3) + 3 = 6 \)

\[
\lim_{{x \to -3^-}} f(x) = \lim_{{x \to -3^-}} (-x + 3) = -(−3) + 3 = 6
\]

\[
\lim_{{x \to -3^+}} f(x) = \lim_{{x \to -3^+}} (-2x) = -2(-3) = 6
\]

\[\therefore \lim_{{x \to -3}} f(x) = f(-3)\]

Therefore, \( f \) is continuous at \( x = -3 \)

**Case III:**

If \( -3 < c < 3 \), then \( f(c) = -2c \) and \( \lim_{{x \to c^-}} f(x) = \lim_{{x \to c^-}} (-2x) = -2c \)

\[\therefore \lim_{{x \to c}} f(x) = f(c)\]

Therefore, \( f \) is continuous in \((-3, 3)\).

**Case IV:**

If \( c = 3 \), then the left hand limit of \( f \) at \( x = 3 \) is,

\[
\lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^-}} (-2x) = -2 \times 3 = -6
\]

The right hand limit of \( f \) at \( x = 3 \) is,

\[
\lim_{{x \to 3^+}} f(x) = \lim_{{x \to 3^+}} (6x + 2) = 6 \times 3 + 2 = 20
\]

It is observed that the left and right hand limit of \( f \) at \( x = 3 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 3 \)

**Case V:**

If \( c > 3 \), then \( f(c) = 6c + 2 \) and \( \lim_{{x \to c^-}} f(x) = \lim_{{x \to c^-}} (6x + 2) = 6c + 2 \)

\[\therefore \lim_{{x \to c}} f(x) = f(c)\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 3 \)

Hence, \( x = 3 \) is the only point of discontinuity of \( f \).

**Question 8:**

Find all points of discontinuity of \( f \), where \( f \) is defined by
\[ f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \]

Answer

\[ f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \]

The given function \( f \) is defined at all the points of the real line. Let \( c \) be a point on the real line.

Case I:

If \( c < 0 \), then \( f(c) = -1 \)

\[ \lim_{x \to c} f(x) = \lim_{x \to c} \left( \frac{|x|}{x} \right) = -1 \text{ if } x < 0 \]

\( f(x) = 0 \) if \( x = 0 \)

\[ \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x < 0 \)

Case II:

If \( c = 0 \), then the left hand limit of \( f \) at \( x = 0 \) is,

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1 \]

The right hand limit of \( f \) at \( x = 0 \) is,

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1 \]

It is observed that the left and right hand limit of \( f \) at \( x = 0 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 0 \)

Case III:
If \( c > 0 \), then \( f(c) = 1 \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 0 \)

Hence, \( x = 0 \) is the only point of discontinuity of \( f \).

**Question 9:**
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
\frac{x}{|x|}, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0
\end{cases}
\]

**Answer**

The given function \( f \) is

It is known that, \( x < 0 \Rightarrow |x| = -x \)

Therefore, the given function can be rewritten as

\[
f(x) = \begin{cases} 
\frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0
\end{cases}
\]

\[
\Rightarrow f(x) = -1 \text{ for all } x \in \mathbb{R}
\]

Let \( c \) be any real number. Then, \( \lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1 \)

Also,

\[
f(c) = -1 = \lim_{x \to c} f(x)
\]

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

**Question 10:**
Find all points of discontinuity of \( f \), where \( f \) is defined by
The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 1$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$

Therefore, $f$ is continuous at all points $x$, such that $x < 1$.

Case II:

If $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$

The left hand limit of $f$ at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2$

The right hand limit of $f$ at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2$

Therefore, $f$ is continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1$

Therefore, $f$ is continuous at all points $x$, such that $x > 1$.

Hence, the given function $f$ has no point of discontinuity.

**Question 11:**

Find all points of discontinuity of $f$, where $f$ is defined by
The given function $f$ is defined at all the points of the real line. Let $c$ be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and \[ \lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3) = c^3 - 3 \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, $f$ is continuous at all points $x$, such that $x < 2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

\[ \lim_{x \to 2} f(x) = \lim_{x \to 2} (x^3 - 3) = 2^3 - 3 = 5 \]

\[ \lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2 + 1) = 2^2 + 1 = 5 \]

\[ \therefore \lim_{x \to 2} f(x) = f(2) \]

Therefore, $f$ is continuous at $x = 2$

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

\[ \lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1 \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, $f$ is continuous at all points $x$, such that $x > 2$

Thus, the given function $f$ is continuous at every point on the real line.

Hence, $f$ has no point of discontinuity.

**Question 12:**

Find all points of discontinuity of $f$, where $f$ is defined by

$\begin{array}{l}
  f(x) = \begin{cases} 
    x^3 - 3, & \text{if } x \leq 2 \\
    x^2 + 1, & \text{if } x > 2 
  \end{cases}
\end{array}$
The given function \( f \) is defined at all the points of the real line. Let \( c \) be a point on the real line.

Case I:
If \( c < 1 \), then \( f(c) = c^{10} - 1 \) and 
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (x^{10} - 1) = c^{10} - 1
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \)

Case II:
If \( c = 1 \), then the left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^{-}} f(x) = \lim_{x \to 1^{-}} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0
\]
The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^{+}} f(x) = \lim_{x \to 1^{+}} (x^{2}) = 1
\]
It is observed that the left and right hand limit of \( f \) at \( x = 1 \) do not coincide.
Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:
If \( c > 1 \), then \( f(c) = c^{2} \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (x^{2}) = c^{2}
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observation, it can be concluded that \( x = 1 \) is the only point of discontinuity of \( f \).

Question 13:
Is the function defined by
\[
f(x) = \begin{cases}  
     x^{10} - 1, & \text{if } x \leq 1 \\
     x^{2}, & \text{if } x > 1
\end{cases}
\]
a continuous function?

Answer

\[ f(x) = \begin{cases} 
  x + 5, & \text{if } x \leq 1 \\
  x - 5, & \text{if } x > 1 
\end{cases} \]

The given function is

The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

Case I:

If \( c < 1 \), then \( f(c) = c + 5 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x + 5) = c + 5 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \)

Case II:

If \( c = 1 \), then \( f(1) = 1 + 5 = 6 \)

The left hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6 \]

The right hand limit of \( f \) at \( x = 1 \) is,

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = 1 - 5 = -4 \]

It is observed that the left and right hand limit of \( f \) at \( x = 1 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:

If \( c > 1 \), then \( f(c) = c - 5 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5 \)

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observation, it can be concluded that \( x = 1 \) is the only point of discontinuity of \( f \).

Question 14:
Discuss the continuity of the function \( f \), where \( f \) is defined by
\[
 f(x) = \begin{cases} 
 3, & \text{if } 0 \leq x \leq 1 \\
 4, & \text{if } 1 < x < 3 \\
 5, & \text{if } 3 \leq x \leq 10 
\end{cases}
\]

Answer
\[
 f(x) = \begin{cases} 
 3, & \text{if } 0 \leq x \leq 1 \\
 4, & \text{if } 1 < x < 3 \\
 5, & \text{if } 3 \leq x \leq 10 
\end{cases}
\]

The given function is defined at all points of the interval \([0, 10]\).

Let \( c \) be a point in the interval \([0, 10]\).

Case I:
If \( 0 \leq c < 1 \), then \( f(c) = 3 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (3) = 3 \)
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous in the interval \([0, 1)\).

Case II:
If \( c = 1 \), then \( f(1) = 3 \)
The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3) = 3
\]
The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (4) = 4
\]
It is observed that the left and right hand limits of \( f \) at \( x = 1 \) do not coincide.
Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:
If \( 1 < c < 3 \), then \( f(c) = 4 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4 \)
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points of the interval \((1, 3)\).

Case IV:
If \( c = 3 \), then \( f(c) = 5 \)
The left hand limit of $f$ at $x = 3$ is,
\[ \lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^-}} (4) = 4 \]
The right hand limit of $f$ at $x = 3$ is,
\[ \lim_{{x \to 3^+}} f(x) = \lim_{{x \to 3^+}} (5) = 5 \]
It is observed that the left and right hand limits of $f$ at $x = 3$ do not coincide.
Therefore, $f$ is not continuous at $x = 3$

Case V:
If $3 < c \leq 10$, then $f(c) = 5$ and \( \lim_{{x \to c}} f(x) = \lim_{{x \to c}} (5) = 5 \)
\[ \lim_{{x \to c}} f(x) = f(c) \]
Therefore, $f$ is continuous at all points of the interval $(3, 10]$. Hence, $f$ is not continuous at $x = 1$ and $x = 3$

**Question 15:**
Discuss the continuity of the function $f$, where $f$ is defined by
\[ f(x) = \begin{cases} 
2x, & \text{if } x < 0 \\
0, & \text{if } 0 \leq x \leq 1 \\
4x, & \text{if } x > 1 
\end{cases} \]

**Answer**
\[ f(x) = \begin{cases} 
2x, & \text{if } x < 0 \\
0, & \text{if } 0 \leq x \leq 1 \\
4x, & \text{if } x > 1 
\end{cases} \]
The given function is defined at all points of the real line.
Let $c$ be a point on the real line.

**Case I:**
If $c < 0$, then $f(c) = 2c$
\[ \lim_{{x \to c}} f(x) = \lim_{{x \to c}} (2x) = 2c \]
\[ \therefore \lim_{{x \to c}} f(x) = f(c) \]
Therefore, $f$ is continuous at all points $x$, such that $x < 0$
Case II:
If \( c = 0 \), then \( f(c) = f(0) = 0 \)

The left hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} (2x) = 2 \times 0 = 0
\]

The right hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} (0) = 0
\]
\[ \therefore \lim_{x \to 0} f(x) = f(0) \]

Therefore, \( f \) is continuous at \( x = 0 \)

Case III:
If \( 0 < c < 1 \), then \( f(x) = 0 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0 \)
\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points of the interval \((0, 1)\).

Case IV:
If \( c = 1 \), then \( f(c) = f(1) = 0 \)

The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} (0) = 0
\]

The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} (4x) = 4 \times 1 = 4
\]

It is observed that the left and right hand limits of \( f \) at \( x = 1 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 1 \)

Case V:
If \( c < 1 \), then \( f(c) = 4c \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c \)
\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Hence, \( f \) is not continuous only at \( x = 1 \)

Question 16:
Discuss the continuity of the function \( f \), where \( f \) is defined by
Answer

\[
f(x) = \begin{cases} 
-2, & \text{if } x \leq -1 \\
2x, & \text{if } -1 < x \leq 1 \\
2, & \text{if } x > 1 
\end{cases}
\]

The given function \( f \) is defined at all points of the real line.

Let \( c \) be a point on the real line.

Case I:

If \( c < -1 \), then \( f(c) = -2 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2 \)

\[\therefore \lim_{x \to c} f(x) = f(c)\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < -1 \)

Case II:

If \( c = -1 \), then \( f(c) = f(-1) = -2 \)

The left hand limit of \( f \) at \( x = -1 \) is,

\[\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2\]

The right hand limit of \( f \) at \( x = -1 \) is,

\[\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2 \times (-1) = -2\]

\[\therefore \lim_{x \to -1} f(x) = f(-1)\]

Therefore, \( f \) is continuous at \( x = -1 \)

Case III:

If \( -1 < c < 1 \), then \( f(c) = 2c \)

\[\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c\]

\[\therefore \lim_{x \to c} f(x) = f(c)\]

Therefore, \( f \) is continuous at all points of the interval \((-1, 1)\).

Case IV:
If \( c = 1 \), then \( f(c) = f(1) = 2 \times 1 = 2 \)

The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2
\]

The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2
\]

\[ \therefore \lim_{x \to 1} f(x) = f(c) \]

Therefore, \( f \) is continuous at \( x = 2 \)

Case V:

If \( c > 1 \), then \( f(c) = 2 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2 \)

\[ \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observations, it can be concluded that \( f \) is continuous at all points of the real line.

**Question 17:**

Find the relationship between \( a \) and \( b \) so that the function \( f \) defined by
\[
f(x) = \begin{cases} 
ax + 1, & \text{if } x \leq 3 \\
bx + 3, & \text{if } x > 3 
\end{cases}
\]
is continuous at \( x = 3 \).

Answer

\[
f(x) = \begin{cases} 
ax + 1, & \text{if } x \leq 3 \\
bx + 3, & \text{if } x > 3 
\end{cases}
\]

The given function \( f \) is

If \( f \) is continuous at \( x = 3 \), then
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (ax + 1) = 3\alpha + 1
\]
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (bx + 3) = 3\beta + 3
\]
\[
f(3) = 3\alpha + 1
\]
Therefore, from (1), we obtain
\[
3\alpha + 1 = 3\beta + 3
\]
\[
\Rightarrow 3\alpha + 1 = 3\beta + 3
\]
\[
\Rightarrow 3\alpha = 3\beta + 2
\]
\[
\Rightarrow \alpha = \beta + \frac{2}{3}
\]
Therefore, the required relationship is given by,
\[
\alpha = \beta + \frac{2}{3}
\]

**Question 18:**

For what value of \( \lambda \) is the function defined by
\[
f(x) = \begin{cases} 
\lambda (x^2 - 2x), & \text{if } x \leq 0 \\
4x + 1, & \text{if } x > 0 
\end{cases}
\]
continuous at \( x = 0 \)? What about continuity at \( x = 1 \)?

**Answer**

The given function \( f \) is

If \( f \) is continuous at \( x = 0 \), then
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)
\]
\[
\Rightarrow \lambda (x^2 - 2x) = \lim_{x \to 0} (4x + 1) = \lambda (0^2 - 2 \times 0)
\]
\[
\Rightarrow \lambda (0^2 - 2 \times 0) = 4 \times 0 + 1 = 0
\]
\[
\Rightarrow 0 = 1 = 0, \text{ which is not possible}
\]

Therefore, there is no value of \( \lambda \) for which \( f \) is continuous at \( x = 0 \).
At $x = 1$,

$$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$$

$$\lim_{x \to 1} (4x + 1) = 4 \times 1 + 1 = 5$$

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

Therefore, for any values of $\lambda$, $f$ is continuous at $x = 1$

**Question 19:**

Show that the function defined by $g(x) = x - \lfloor x \rfloor$ is discontinuous at all integral point.

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

**Answer**

The given function is $g(x) = x - \lfloor x \rfloor$

It is evident that $g$ is defined at all integral points.

Let $n$ be an integer.

Then,

$$g(n) = n - \lfloor n \rfloor = n - n = 0$$

The left hand limit of $f$ at $x = n$ is,

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-} (x - \lfloor x \rfloor) = \lim_{x \to n^-} x - \lim_{x \to n^-} \lfloor x \rfloor = n - (n - 1) = 1$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+} (x - \lfloor x \rfloor) = \lim_{x \to n^+} x - \lim_{x \to n^+} \lfloor x \rfloor = n - n = 0$$

It is observed that the left and right hand limits of $f$ at $x = n$ do not coincide.

Therefore, $f$ is not continuous at $x = n$

Hence, $g$ is discontinuous at all integral points.

**Question 20:**

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = p$?

**Answer**

The given function is $f(x) = x^2 - \sin x + 5$
It is evident that \( f \) is defined at \( x = \pi \)

At \( x = \pi, \ f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5 \)

Consider \( \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \)

Put \( x = \pi + h \)

If \( x \to \pi \), then it is evident that \( h \to 0 \)

\[
\therefore \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5)
\]

\[
= \lim_{x \to 0} \left[ (\pi + h)^2 - \sin (\pi + h) + 5 \right]
\]

\[
= \lim_{x \to 0} (\pi + h)^2 - \lim_{x \to 0} \sin (\pi + h) + \lim_{x \to 0} 5
\]

\[
= (\pi + 0)^2 - 0 + 5
\]

\[
= \pi^2 - \sin \pi + 5
\]

\[
= \pi^2 - 0\times1 - (-1)\times0 + 5
\]

\[
= \pi^2 + 5
\]

\[
\therefore \lim_{x \to \pi} f(x) = f(\pi)
\]

Therefore, the given function \( f \) is continuous at \( x = \pi \)

**Question 21:**

Discuss the continuity of the following functions.

(a) \( f(x) = \sin x + \cos x \)

(b) \( f(x) = \sin x - \cos x \)

(c) \( f(x) = \sin x \times \cos x \)

**Answer**

It is known that if \( g \) and \( h \) are two continuous functions, then

\( g + h, \ g - h, \) and \( gh \) are also continuous.

It has to be proved first that \( g(x) = \sin x \) and \( h(x) = \cos x \) are continuous functions.

Let \( g(x) = \sin x \)

It is evident that \( g(x) = \sin x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)
Therefore, $g$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

\[
\lim_{x \to c} h(x) = \lim_{x \to c} \cos x
= \lim_{h \to 0} \cos(c + h)
= \lim_{h \to 0} \cos(c \cos h - \sin c \sin h)
= \lim_{h \to 0} \cos(c \cos h) \cdot \lim_{h \to 0} \cos \sin h
= \cos c \cdot 1 \cdot \cos 0
= \cos c
\]

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, $h$ is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function
Question 22:
Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer
It is known that if \( g \) and \( h \) are two continuous functions, then

\[
(i) \quad \frac{h(x)}{g(x)} , \ g(x) \neq 0 \text{ is continuous}
\]

\[
(ii) \quad \frac{1}{g(x)} , \ g(x) \neq 0 \text{ is continuous}
\]

\[
(iii) \quad \frac{1}{h(x)} , \ h(x) \neq 0 \text{ is continuous}
\]

It has to be proved first that \( g (x) = \sin x \) and \( h (x) = \cos x \) are continuous functions.
Let \( g (x) = \sin x \)
It is evident that \( g (x) = \sin x \) is defined for every real number.
Let \( c \) be a real number. Put \( x = c + h \)
If \( x \to c \), then \( h \to 0 \)

\[
g(c) = \sin c
\]

\[
\lim_{x \to c} g(x) = \lim_{x \to c} \sin x = \lim_{h \to 0} \sin (c+h) = \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] = \sin c \cos 0 + \cos c \sin 0 = \sin c + 0 = \sin c
\]

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is a continuous function.
Let \( h (x) = \cos x \)
It is evident that \( h (x) = \cos x \) is defined for every real number.
Let \( c \) be a real number. Put \( x = c + h \)
If \( x \to c \), then \( h \to 0 \)

\[
h(c) = \cos c
\]
\[
\lim_{x \to c} h(x) = \lim_{x \to c} \cos x \\
= \lim_{h \to 0} \cos(c + h) \\
= \lim_{h \to 0} \left[ \cos c \cos h - \sin c \sin h \right] \\
= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
= \cos c \cos 0 - \sin c \sin 0 \\
= \cos c \times 1 - \sin c \times 0 \\
= \cos c
\]

\[\therefore \lim_{x \to c} h(x) = h(c)\]

Therefore, \( h(x) = \cos x \) is a continuous function.

It can be concluded that,

\[\csc x = \frac{1}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}\]

\[\Rightarrow \csc x, \quad x \neq n\pi \quad (n \in \mathbb{Z}) \text{ is continuous}\]

Therefore, cosecant is continuous except at \( x = np, \; n \in \mathbb{Z} \)

\[\sec x = \frac{1}{\cos x}, \quad \cos x \neq 0 \text{ is continuous}\]

\[\Rightarrow \sec x, \quad x \neq (2n + 1)\frac{\pi}{2} \quad (n \in \mathbb{Z}) \text{ is continuous}\]

Therefore, secant is continuous except at \( x = (2n + 1)\frac{\pi}{2}, \; (n \in \mathbb{Z}) \)

\[\cot x = \frac{\cos x}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}\]

\[\Rightarrow \cot x, \quad x \neq n\pi \quad (n \in \mathbb{Z}) \text{ is continuous}\]

Therefore, cotangent is continuous except at \( x = np, \; n \in \mathbb{Z} \)

**Question 23:**

Find the points of discontinuity of \( f \), where

\[f(x) = \begin{cases} 
\frac{\sin x}{x}, & \text{if } x < 0 \\
x + 1, & \text{if } x \geq 0
\end{cases}\]
Answer

\[ f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases} \]

The given function \( f \) is defined at all points of the real line. Let \( c \) be a real number.

Case I:

If \( c < 0 \), then \( f(c) = \frac{\sin c}{c} \) and

\[ \lim_{x \to c} f(x) = \lim_{x \to c} \left( \frac{\sin x}{x} \right) = \frac{\sin c}{c} \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 0 \)

Case II:

If \( c > 0 \), then \( f(c) = c + 1 \) and

\[ \lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1 \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 0 \)

Case III:

If \( c = 0 \), then \( f(c) = f(0) = 0 + 1 = 1 \)

The left hand limit of \( f \) at \( x = 0 \) is,

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} - 1 \]

The right hand limit of \( f \) at \( x = 0 \) is,

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 1 \]

\[ \therefore \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = f(0) \]

Therefore, \( f \) is continuous at \( x = 0 \)

From the above observations, it can be concluded that \( f \) is continuous at all points of the real line.

Thus, \( f \) has no point of discontinuity.
Question 24:
Determine if \( f \) defined by
\[
    f(x) = \begin{cases} 
        x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\
        0, & \text{if } x = 0 
    \end{cases}
\]
is a continuous function?

Answer
\[
    f(x) = \begin{cases} 
        x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\
        0, & \text{if } x = 0 
    \end{cases}
\]
The given function \( f \) is defined at all points of the real line. Let \( c \) be a real number.

Case I:
If \( c \neq 0 \), then \( f(c) = c^2 \sin \frac{1}{c} \).

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \left( x^2 \sin \frac{1}{x} \right) = \left( \lim_{x \to c} x^2 \right) \left( \lim_{x \to c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \neq 0 \).

Case II:
If \( c = 0 \), then \( f(0) = 0 \).
Therefore, \( f \) is continuous at \( x = 0 \).

From the above observations, it can be concluded that \( f \) is continuous at every point of the real line.

Thus, \( f \) is a continuous function.

**Question 25:**

Examine the continuity of \( f \), where \( f \) is defined by

\[
 f(x) = \begin{cases} 
 \sin x - \cos x, & \text{if } x \neq 0 \\
 -1 & \text{if } x = 0 
\end{cases}
\]

**Answer**

\[
 f(x) = \begin{cases} 
 \sin x - \cos x, & \text{if } x \neq 0 \\
 -1 & \text{if } x = 0 
\end{cases}
\]

The given function \( f \) is

It is evident that \( f \) is defined at all points of the real line.

Let \( c \) be a real number.

**Case I:**
If \( c \neq 0 \), then 
\[
\lim_{x \to c} f(x) = \sin c - \cos c
\]
\[
\lim_{x \to c} \left( \sin x - \cos x \right) = \sin c - \cos c
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x \neq 0 \)

Case II:
If \( c = 0 \), then \( f(0) = -1 \)

\[
\lim_{x \to 0^+} \left( \sin x - \cos x \right) = \sin 0 - \cos 0 = 0 - 1 = -1
\]
\[
\lim_{x \to 0^-} \left( \sin x - \cos x \right) = \sin 0 - \cos 0 = 0 - 1 = -1
\]
\[
\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = f(0)
\]
Therefore, \( f \) is continuous at \( x = 0 \)

From the above observations, it can be concluded that \( f \) is continuous at every point of the real line.

Thus, \( f \) is a continuous function.

**Question 26:**
Find the values of \( k \) so that the function \( f \) is continuous at the indicated point.

\[
f(x) = \begin{cases} 
  \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\
  3, & \text{if } x = \frac{\pi}{2}
\end{cases}
\]

**Answer**
\[
f(x) = \begin{cases} 
  \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\
  3, & \text{if } x = \frac{\pi}{2}
\end{cases}
\]

The given function \( f \) is

The given function \( f \) is continuous at \( x = \frac{\pi}{2} \), if \( f \) is defined at \( x = \frac{\pi}{2} \) and if the value of the \( f \) at \( x = \frac{\pi}{2} \) equals the limit of \( f \) at \( x = \frac{\pi}{2} \).
It is evident that $f$ is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

Put $x = \frac{\pi}{2} + h$

Then, $x \to \frac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k \cos \left(\frac{\pi}{2} + h\right)}{\frac{\pi}{2} + h - \frac{\pi}{2}} = \lim_{h \to 0} \frac{k \cos \left(\frac{\pi}{2} + h\right)}{-2h} = \lim_{h \to 0} \frac{k \cos \left(\frac{\pi}{2} + h\right)}{-2}\left(\frac{\sin h}{h}\right) = \frac{k}{2} = \frac{k}{2} = \frac{k}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the required value of $k$ is 6.

**Question 27:**

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x) = \begin{cases} 
  kx^2, & \text{if } x \leq 2 \\
  3, & \text{if } x > 2 
\end{cases}$$

at $x = 2$

**Answer**

$$f(x) = \begin{cases} 
  kx^2, & \text{if } x \leq 2 \\
  3, & \text{if } x > 2 
\end{cases}$$

The given function is

The given function $f$ is continuous at $x = 2$, if $f$ is defined at $x = 2$ and if the value of $f$ at $x = 2$ equals the limit of $f$ at $x = 2$

It is evident that $f$ is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} f(x) = f(2)
\]
\[
\Rightarrow \lim_{x \to 2} (kx^2) = \lim_{x \to 2} (3) = 4k
\]
\[
\Rightarrow k \times 2^2 = 3 = 4k
\]
\[
\Rightarrow 4k = 3
\]
\[
\Rightarrow k = \frac{3}{4}
\]
Therefore, the required value of \(k\) is \(\frac{3}{4}\).

**Question 28:**

Find the values of \(k\) so that the function \(f\) is continuous at the indicated point.

\[
f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq \pi \\
  \cos x, & \text{if } x > \pi 
\end{cases}
\]  

**at \(x = \pi\)**

**Answer**

The given function is

\[
f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq \pi \\
  \cos x, & \text{if } x > \pi 
\end{cases}
\]

The given function \(f\) is continuous at \(x = p\), if \(f\) is defined at \(x = p\) and if the value of \(f\) at \(x = p\) equals the limit of \(f\) at \(x = p\).

It is evident that \(f\) is defined at \(x = p\) and \(f(\pi) = k\pi + 1\)

\[
\lim_{x \to \pi} f(x) = \lim_{x \to \pi} f(x) = f(\pi)
\]
\[
\Rightarrow \lim_{x \to \pi} (kx + 1) = \lim_{x \to \pi} \cos x = k\pi + 1
\]
\[
\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1
\]
\[
\Rightarrow k\pi + 1 = -1 = k\pi + 1
\]
\[
\Rightarrow k = -\frac{2}{\pi}
\]
Therefore, the required value of \(k\) is \(-\frac{2}{\pi}\).
Question 29:
Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x) = \begin{cases} 
  kx + 1, & \text{if } x \leq 5 \\
  3x - 5, & \text{if } x > 5
\end{cases} \quad \text{at } x = 5$$

Answer

The given function $f$ is

The given function $f$ is continuous at $x = 5$, if $f$ is defined at $x = 5$ and if the value of $f$ at $x = 5$ equals the limit of $f$ at $x = 5$

It is evident that $f$ is defined at $x = 5$ and $f(5) = k(5) + 1 = 5k + 1$

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to 5} (kx + 1) = \lim_{x \to 5} (3x - 5) = 5k + 1$$

$$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$$

$$\Rightarrow 5k + 1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

Therefore, the required value of $k$ is $\frac{9}{5}$.

Question 30:
Find the values of $a$ and $b$ such that the function defined by

$$f(x) = \begin{cases} 
  5, & \text{if } x \leq 2 \\
  ax + b, & \text{if } 2 < x < 10 \\
  21, & \text{if } x \geq 10
\end{cases}$$

is a continuous function.
Answer

\[
f(x) = \begin{cases} 
5, & \text{if } x \leq 2 \\
ax + b, & \text{if } 2 < x < 10 \\
21, & \text{if } x \geq 10 
\end{cases}
\]

The given function \( f \) is

It is evident that the given function \( f \) is defined at all points of the real line.

If \( f \) is a continuous function, then \( f \) is continuous at all real numbers.

In particular, \( f \) is continuous at \( x = 2 \) and \( x = 10 \)

Since \( f \) is continuous at \( x = 2 \), we obtain

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)
\]

\[
\Rightarrow \lim_{x \to 2} (5) = \lim_{x \to 2} (ax + b) = 5
\]

\[
\Rightarrow 5 = 2a + b = 5
\]

\[
\Rightarrow 2a + b = 5 \quad \text{...(1)}
\]

Since \( f \) is continuous at \( x = 10 \), we obtain

\[
\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)
\]

\[
\Rightarrow \lim_{x \to 10} (ax + b) = \lim_{x \to 10} (21) = 21
\]

\[
\Rightarrow 10a + b = 21 = 21
\]

\[
\Rightarrow 10a + b = 21 \quad \text{...(2)}
\]

On subtracting equation (1) from equation (2), we obtain

\[
8a = 16
\]

\[
\Rightarrow a = 2
\]

By putting \( a = 2 \) in equation (1), we obtain

\[
2 \times 2 + b = 5
\]

\[
\Rightarrow 4 + b = 5
\]
\[ b = 1 \]

Therefore, the values of \( a \) and \( b \) for which \( f \) is a continuous function are 2 and 1 respectively.

**Question 31:**
Show that the function defined by \( f(x) = \cos(x^2) \) is a continuous function.

**Answer**
The given function is \( f(x) = \cos(x^2) \)
This function \( f \) is defined for every real number and \( f \) can be written as the composition of two functions as,
\[
f = g \circ h, \text{ where } g(x) = \cos x \text{ and } h(x) = x^2.
\]
\[
\therefore (g \circ h)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x).
\]
It has to be first proved that \( g(x) = \cos x \) and \( h(x) = x^2 \) are continuous functions.
It is evident that \( g \) is defined for every real number.
Let \( c \) be a real number.
Then, \( g(c) = \cos c \)
Put \( x = c + h \)
If \( x \to c \), then \( h \to 0 \)
\[
\lim_{x \to c} g(x) = \lim_{x \to c} \cos x
\]
\[
= \lim_{h \to 0} \cos(c + h)
\]
\[
= \lim_{h \to 0} \left[ \cos c \cos h - \sin c \sin h \right]
\]
\[
= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h
\]
\[
= \cos c \times 1 - \sin c \times 0
\]
\[
= \cos c
\]
\[
\therefore \lim_{x \to c} g(x) = g(c)
\]
Therefore, \( g(x) = \cos x \) is a continuous function.
\( h(x) = x^2 \)

Clearly, \( h \) is defined for every real number.

Let \( k \) be a real number, then \( h(k) = k^2 \)

\[
\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2
\]

\[
\therefore \lim_{x \to k} h(x) = h(k)
\]

Therefore, \( h \) is a continuous function.

It is known that for real valued functions \( g \) and \( h \), such that \((g \circ h)\) is defined at \( c \), if \( g \) is continuous at \( c \) and if \( f \) is continuous at \( g(c) \), then \((f \circ g)\) is continuous at \( c \).

Therefore, \( f(x) = (g \circ h)(x) = \cos(x^2) \) is a continuous function.

**Question 32:**

Show that the function defined by \( f(x) = |\cos x| \) is a continuous function.

**Answer**

The given function is \( f(x) = |\cos x| \)

This function \( f \) is defined for every real number and \( f \) can be written as the composition of two functions as,

\( f = g \circ h \), where \( g(x) = x \) and \( h(x) = \cos x \)

\[
\therefore (g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)
\]

It has to be first proved that \( g(x) = |x| \) and \( h(x) = \cos x \) are continuous functions.

\( g(x) = |x| \) can be written as

\[
g(x) = \begin{cases} 
-x, & \text{if } x < 0 \\
x, & \text{if } x \geq 0
\end{cases}
\]

Clearly, \( g \) is defined for all real numbers.

Let \( c \) be a real number.

**Case I:**

If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]
Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

Case II:

If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)

\[ \therefore \lim_{x \to c} g(x) = g(c) \]

Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

Case III:

If \( c = 0 \), then \( g(c) = g(0) = 0 \)

\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0 \]

\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = 0 \]

\[ \therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = g(0) \]

Therefore, \( g \) is continuous at \( x = 0 \)

From the above three observations, it can be concluded that \( g \) is continuous at all points.

\( h(x) = \cos x \)

It is evident that \( h(x) = \cos x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\( h(c) = \cos c \)

\[ \lim_{x \to c} h(x) = \lim_{x \to c} \cos x \]

\[ = \lim_{h \to 0} [\cos(c + h)] \]

\[ = \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \]

\[ = \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \]

\[ = \cos c \cos 0 - \sin c \sin 0 \]

\[ = \cos c \times 1 - \sin c \times 0 \]

\[ = \cos c \]

\[ \therefore \lim_{x \to c} h(x) = h(c) \]

Therefore, \( h(x) = \cos x \) is a continuous function.

It is known that for real valued functions \( g \) and \( h \), such that \( (g \circ h) \) is defined at \( c \), if \( g \) is continuous at \( c \) and if \( f \) is continuous at \( g(c) \), then \((f \circ g)\) is continuous at \( c \).
Therefore, \( f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| \) is a continuous function.

**Question 33:**

Examine that \( \sin |x| \) is a continuous function.

**Answer**

Let \( f(x) = \sin |x| \)

This function \( f \) is defined for every real number and \( f \) can be written as the composition of two functions as,

\[
f = g \circ h, \quad \text{where} \quad g(x) = |x| \quad \text{and} \quad h(x) = \sin x
\]

\[
\therefore (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)
\]

It has to be proved first that \( g(x) = |x| \) and \( h(x) = \sin x \) are continuous functions.

\( g(x) = |x| \) can be written as

\[
g(x) = \begin{cases} 
-x, & \text{if } x < 0 \\
x, & \text{if } x \geq 0 
\end{cases}
\]

Clearly, \( g \) is defined for all real numbers.

Let \( c \) be a real number.

**Case I:**

If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

**Case II:**

If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

**Case III:**

If \( c = 0 \), then \( g(c) = g(0) = 0 \)
Therefore, $g$ is continuous at $x = 0$

From the above three observations, it can be concluded that $g$ is continuous at all points.

$h (x) = \sin x$

It is evident that $h (x) = \sin x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h (c) = \sin c$

\[ h(c) = \sin c \]

\[ \lim_{x \to c} h(x) = \lim_{x \to c} \sin x \]

\[ = \lim_{k \to 0} \sin(c + k) \]

\[ = \lim_{k \to 0} [\sin c \cos k + \cos c \sin k] \]

\[ = \lim_{k \to 0} (\sin c \cos k) + \lim_{k \to 0} (\cos c \sin k) \]

\[ = \sin c \cos 0 + \cos c \sin 0 \]

\[ = \sin c \]

\[ \therefore \lim_{x \to c} h(x) = g(c) \]

Therefore, $h$ is a continuous function.

It is known that for real valued functions $g$ and $h$, such that $(g \circ h)$ is defined at $c$, if $g$ is continuous at $c$ and if $f$ is continuous at $g(c)$, then $(f \circ g)$ is continuous at $c$.

Therefore, \( f(x) = (g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x| \) is a continuous function.

**Question 34:**

Find all the points of discontinuity of $f$ defined by $f(x) = |x| - |x + 1|$. 

**Answer**

The given function is $f(x) = |x| - |x + 1|$
The two functions, \( g \) and \( h \), are defined as
\[
g(x) = |x| \quad \text{and} \quad h(x) = |x + 1|
\]

Then, \( f = g - h \)

The continuity of \( g \) and \( h \) is examined first.
\[
g(x) = |x| \quad \text{can be written as}
\]
\[
g(x) = \begin{cases} 
-x, & \text{if } x < 0 \\
0, & \text{if } x = 0 \\
x, & \text{if } x > 0
\end{cases}
\]

Clearly, \( g \) is defined for all real numbers.

Let \( c \) be a real number.

Case I:
If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)
\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \).

Case II:
If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)
\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \).

Case III:
If \( c = 0 \), then \( g(c) = g(0) = 0 \)
\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0
\]
\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} x = 0
\]
\[
\therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} x = g(0)
\]

Therefore, \( g \) is continuous at \( x = 0 \)

From the above three observations, it can be concluded that \( g \) is continuous at all points.

\[
h(x) = |x + 1| \quad \text{can be written as}
\]
\[
h(x) = \begin{cases} 
-(x + 1), & \text{if } x < -1 \\
x + 1, & \text{if } x \geq -1
\end{cases}
\]

Clearly, \( h \) is defined for every real number.
Let \( c \) be a real number.

Case I:
If \( c < -1 \), then \( h(c) = -(c+1) \) and 
\[
\lim_{x \to c} h(x) = \lim_{x \to c} [-x+1] = -(c+1)
\]
\[
\therefore \lim_{x \to c} h(x) = h(c)
\]
Therefore, \( h \) is continuous at all points \( x \), such that \( x < -1 \)

Case II:
If \( c > -1 \), then \( h(c) = c+1 \) and 
\[
\lim_{x \to c} h(x) = \lim_{x \to c} (x+1) = c+1
\]
\[
\therefore \lim_{x \to c} h(x) = h(c)
\]
Therefore, \( h \) is continuous at all points \( x \), such that \( x > -1 \)

Case III:
If \( c = -1 \), then \( h(c) = h(-1) = -1+1 = 0 \)
\[
\lim_{x \to -1} h(x) = \lim_{x \to -1} [-x+1] = -(-1+1) = 0
\]
\[
\lim_{x \to -1} h(x) = \lim_{x \to -1} (x+1) = (-1+1) = 0
\]
\[
\therefore \lim_{x \to -1} h(x) = \lim_{h \to -1} h(x) = h(-1)
\]
Therefore, \( h \) is continuous at \( x = -1 \)

From the above three observations, it can be concluded that \( h \) is continuous at all points of the real line.

\( g \) and \( h \) are continuous functions. Therefore, \( f = g - h \) is also a continuous function.

Therefore, \( f \) has no point of discontinuity.
Exercise 5.2

Question 1:
Differentiate the functions with respect to $x$.

$$\sin(x^2 + 5)$$

Answer

Let $f(x) = \sin(x^2 + 5)$, $u(x) = x^2 + 5$, and $v(t) = \sin t$

Then, $(v \circ u)(x) = v(u(x)) = v(x^2 + 5) = \tan(x^2 + 5) = f(x)$

Thus, $f$ is a composite of two functions.

Put $t = u(x) = x^2 + 5$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

Therefore, by chain rule, $
\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \cdot 2x = 2x \cos(x^2 + 5)$

Alternate method

$$\frac{d}{dx} \left[ \sin(x^2 + 5) \right] = \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5)$$

$$= \cos(x^2 + 5) \cdot \left[ \frac{d}{dx}(x^2) + \frac{d}{dx}(5) \right]$$

$$= \cos(x^2 + 5) \cdot [2x + 0]$$

$$= 2x \cos(x^2 + 5)$$

Question 2:
Differentiate the functions with respect to $x$.

$$\cos(\sin x)$$
Answer

Let \( f(x) = \cos(\sin x), u(x) = \sin x \), and \( v(t) = \cos t \)

Then, \( (vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x) \)

Thus, \( f \) is a composite function of two functions.

Put \( t = u(x) = \sin x \)

\[
\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)
\]

\[
\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x
\]

By chain rule, \( \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x) \)

Alternate method

\[
\frac{d}{dx}[\cos(\sin x)] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)
\]

Question 3:

Differentiate the functions with respect to \( x \).

\( \sin(ax + b) \)

Answer

Let \( f(x) = \sin(ax + b), u(x) = ax + b \), and \( v(t) = \sin t \)

Then, \( (vou)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = f(x) \)

Thus, \( f \) is a composite function of two functions, \( u \) and \( v \).

Put \( t = u(x) = ax + b \)

Therefore,

\[
\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)
\]

\[
\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a
\]

Hence, by chain rule, we obtain

\[
\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)
\]
Alternate method
\[
\frac{d}{dx} [\sin(ax + b)] = \cos(ax + b) \cdot \frac{d}{dx} (ax + b)
\]
\[
= \cos(ax + b) \left[ \frac{d}{dx}(ax) + \frac{d}{dx}(b) \right]
\]
\[
= \cos(ax + b)(a + 0)
\]
\[
= a \cos(ax + b)
\]

Question 4:
Differentiate the functions with respect to \(x\).
\[\sec\left(\tan\left(\sqrt{x}\right)\right)\]

Answer
Let \(f(x) = \sec\left(\tan\sqrt{x}\right), u(x) = \sqrt{x}, v(i) = \tan i, \) and \(w(s) = \sec s\)

Then, \((wovou)(x) = w[v(u(x))] = w(\sqrt{x}) = w(\tan \sqrt{x}) = \sec(\tan \sqrt{x}) = f(x)\)

Thus, \(f\) is a composite function of three functions, \(u, v,\) and \(w\).

Put \(s = v(i) = \tan t\) and \(t = u(x) = \sqrt{x}\)

Then, \(\frac{dw}{ds} = \frac{d}{ds} [\sec s] = \sec s \tan s = \sec(\tan t) \cdot \tan(\tan t)\) \[s = \tan t\]

\[
= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x})\] \[t = \sqrt{x}\]

\[
\frac{ds}{dt} = \frac{d}{dt} (\tan t) = \sec^2 t = \sec^2 \sqrt{x}
\]

\[
\frac{dt}{dx} = \frac{d}{dx} (\sqrt{x}) = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}
\]

Hence, by chain rule, we obtain
Alternate method

\[
\frac{d}{dx} \left[ \sec \left( \tan \sqrt{x} \right) \right] = \sec \left( \tan \sqrt{x} \right) \cdot \tan \left( \tan \sqrt{x} \right) \cdot \sec^3 \left( \sqrt{x} \right) \cdot \frac{d}{dx} \left( \sqrt{x} \right)
\]

\[
= \sec \left( \tan \sqrt{x} \right) \cdot \tan \left( \tan \sqrt{x} \right) \cdot \sec^3 \left( \sqrt{x} \right) \cdot \frac{1}{2\sqrt{x}}
\]

\[
= \frac{\sec \left( \tan \sqrt{x} \right) \cdot \tan \left( \tan \sqrt{x} \right) \cdot \sec^2 \left( \sqrt{x} \right)}{2\sqrt{x}}
\]

**Question 5:**

Differentiate the functions with respect to \( x \).

\[
\frac{\sin (ax + b)}{\cos (cx + d)}
\]

**Answer**

\[
f(x) = \frac{\sin (ax + b)}{\cos (cx + d)} = \frac{g(x)}{h(x)}
\]

The given function is

\[
g(x) = \sin (ax + b)
\]

\[
h(x) = \cos (cx + d)
\]

\[
\therefore f' = \frac{g'h - gh'}{h^2}
\]

Consider \( g(x) = \sin (ax + b) \)

Let \( u(x) = ax + b, v(t) = \sin t \)

Then, \( (vu)(x) = v(u(x)) = v(ax + b) = \sin (ax + b) = g(x) \)
∴ \( g \) is a composite function of two functions, \( u \) and \( v \).

Put \( t = u(x) = ax + b \)
\[
\frac{dv}{dt} = \frac{d}{dt} (\sin t) = \cos t = \cos (ax + b)
\]
\[
\frac{dt}{dx} = \frac{d}{dx} (ax + b) = \frac{d}{dx} (ax) + \frac{d}{dx} (b) = a + 0 = a
\]

Therefore, by chain rule, we obtain
\[
\frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos (ax + b) \cdot a = a \cos (ax + b)
\]

Consider \( h(x) = \cos (cx + d) \)

Let \( p(x) = cx + d \), \( q(y) = \cos y \)

Then, \( (q \circ p)(x) = q(p(x)) = q(cx + d) = \cos (cx + d) = h(x) \)

∴ \( h \) is a composite function of two functions, \( p \) and \( q \).

Put \( y = p(x) = cx + d \)
\[
\frac{dq}{dy} = \frac{d}{dy} (\cos y) = -\sin y = -\sin (cx + d)
\]
\[
\frac{dy}{dx} = \frac{d}{dx} (cx + d) = \frac{d}{dx} (cx) + \frac{d}{dx} (d) = c
\]

Therefore, by chain rule, we obtain
\[
\frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin (cx + d) \times c = -c \sin (cx + d)
\]
Question 6:
Differentiate the functions with respect to \( x \).

\[ \cos^2 x \cdot \sin^2 \left( x^3 \right) \]

Answer

The given function is \( \cos^2 x \cdot \sin^2 \left( x^3 \right) \).

\[
\frac{d}{dx} \left[ \cos^2 x \cdot \sin^2 \left( x^3 \right) \right] = \sin^2 \left( x^3 \right) \cdot \frac{d}{dx} \left( \cos x^3 \right) + \cos x^3 \cdot \frac{d}{dx} \left[ \sin^2 \left( x^3 \right) \right]
\]

\[ = \sin^2 \left( x^3 \right) \cdot \left( -\sin x^3 \right) \cdot 3x^2 + \cos x^3 \cdot 2 \sin x^3 \cdot \frac{d}{dx} \left[ \sin x^3 \right]
\]

\[ = -3x^2 \sin x^3 \cdot \sin^2 \left( x^3 \right) + 2 \sin x^3 \cos x^3 \cdot \cos x^3 \cdot 3x^2
\]

\[ = 10x^4 \sin x^3 \cos x^3 - 3x^2 \sin x^3 \sin^2 \left( x^3 \right)
\]

Question 7:
Differentiate the functions with respect to \( x \).

\[ 2 \sqrt{\cot \left( x^2 \right)} \]

Answer
\[ \frac{d}{dx} \left[ 2 \sqrt{\cot(x^2)} \right] \]
\[ = 2 \cdot \frac{1}{2 \sqrt{\cot(x^2)}} \times \frac{d}{dx} \left[ \cot(x^2) \right] \]
\[ = \frac{\sin(x^2)}{\cos(x^2)} \times \csc^2(x^2) \times \frac{d}{dx}(x^2) \]
\[ = -\frac{\sin(x^2)}{\cos(x^2)} \times \frac{1}{\sin^2(x^2)} \times (2x) \]
\[ = \frac{-2x}{\sqrt{\cos x^2 \sin x^2}} \]
\[ = \frac{-2\sqrt{2}x}{\sqrt{2 \sin x^2 \cos x^2 \sin x^2}} \]
\[ = \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}} \]

Question 8:
Differentiate the functions with respect to \( x \).

\[ \cos(\sqrt{x}) \]

Answer
Let \( f(x) = \cos(\sqrt{x}) \)
Also, let \( u(x) = \sqrt{x} \)
And, \( v(t) = \cos t \)
Then, \( (v \circ u)(x) = v(u(x)) \)
\[ = v(\sqrt{x}) \]
\[ = \cos \sqrt{x} \]
\[ = f(x) \]

Clearly, \( f \) is a composite function of two functions, \( u \) and \( v \), such that
\[ t = u(x) = \sqrt{x} \]
Then, \( \frac{dt}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{1}{2 x^{-\frac{3}{2}}} \)

\[ = \frac{1}{2\sqrt{x}} \]

And, \( \frac{dy}{dt} = \frac{d}{dt}(\cos t) = -\sin t \)

\[ = -\sin \left(\sqrt{x}\right) \]

By using chain rule, we obtain

\[ \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \]

\[ = -\sin \left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} \]

\[ = -\frac{1}{2\sqrt{x}} \cdot \sin \left(\sqrt{x}\right) \]

\[ = -\frac{\sin \left(\sqrt{x}\right)}{2\sqrt{x}} \]

**Alternate method**

\[ \frac{d}{dx}\left[ \cos \left(\sqrt{x}\right) \right] = -\sin \left(\sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right) \]

\[ = -\sin \left(\sqrt{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x^2}\right) \]

\[ = -\sin \sqrt{x} \cdot 2 x^{-\frac{3}{2}} \]

\[ = -\frac{\sin \sqrt{x}}{2\sqrt{x}} \]

**Question 9:**

Prove that the function \( f \) given by \( f(x) = |x-1|, x \in \mathbb{R} \) is not differentiable at \( x = 1 \).

**Answer**

The given function is \( f(x) = |x-1|, x \in \mathbb{R} \)
It is known that a function \( f \) is differentiable at a point \( x = c \) in its domain if both

\[
\lim_{{h \to 0^-}} \frac{f(c + h) - f(c)}{h} \quad \text{and} \quad \lim_{{h \to 0^+}} \frac{f(c + h) - f(c)}{h}
\]

are finite and equal.

To check the differentiability of the given function at \( x = 1 \), consider the left hand limit of \( f \) at \( x = 1 \)

\[
\lim_{{h \to 0^-}} \frac{f(1 + h) - f(1)}{h} = \lim_{{h \to 0^-}} \frac{|1 + h - 1| - |1 - 1|}{h} = \lim_{{h \to 0^-}} \frac{-h}{h} = -1
\]

Consider the right hand limit of \( f' \) at \( x = 1 \)

\[
\lim_{{h \to 0^+}} \frac{f(1 + h) - f(1)}{h} = \lim_{{h \to 0^+}} \frac{|1 + h - 1| - |1 - 1|}{h} = \lim_{{h \to 0^+}} \frac{h}{h} = 1
\]

Since the left and right hand limits of \( f \) at \( x = 1 \) are not equal, \( f \) is not differentiable at \( x = 1 \).

**Question 10:**

Prove that the greatest integer function defined by \( f(x) = \lfloor x \rfloor, 0 < x < 3 \) is not differentiable at \( x = 1 \) and \( x = 2 \).

**Answer**

The given function \( f \) is \( f(x) = \lfloor x \rfloor, 0 < x < 3 \)

It is known that a function \( f \) is differentiable at a point \( x = c \) in its domain if both

\[
\lim_{{h \to 0^-}} \frac{f(c + h) - f(c)}{h} \quad \text{and} \quad \lim_{{h \to 0^+}} \frac{f(c + h) - f(c)}{h}
\]

are finite and equal.

To check the differentiability of the given function at \( x = 1 \), consider the left hand limit of \( f \) at \( x = 1 \)
Since the left and right hand limits of \( f \) at \( x = 1 \) are not equal, \( f \) is not differentiable at \( x = 1 \).

To check the differentiability of the given function at \( x = 2 \), consider the left hand limit of \( f \) at \( x = 2 \)

\[
\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[2+h] - [2]}{h} = \lim_{h \to 0^-} \frac{1}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty
\]

Consider the right hand limit of \( f \) at \( x = 2 \)

\[
\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h} = \lim_{h \to 0^+} \frac{2}{h} = \lim_{h \to 0^+} \frac{-2}{h} = 0 = 0
\]

Since the left and right hand limits of \( f \) at \( x = 2 \) are not equal, \( f \) is not differentiable at \( x = 2 \).
Exercise 5.3

Question 1:

\[ \frac{dy}{dx} \]

Find \( \frac{dx}{dx} : \)

\( 2x + 3y = \sin x \)

Answer

The given relationship is \( 2x + 3y = \sin x \)

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x) \]

\[ \Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x \]

\[ \Rightarrow 2 + 3 \frac{dy}{dx} = \cos x \]

\[ \Rightarrow 3 \frac{dy}{dx} = \cos x - 2 \]

\[ \therefore \frac{dy}{dx} = \frac{\cos x - 2}{3} \]

Question 2:

\[ \frac{dy}{dx} \]

Find \( \frac{dx}{dx} : \)

\( 2x + 3y = \sin y \)

Answer

The given relationship is \( 2x + 3y = \sin y \)

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y) \]
Question 3:

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\( ax + by^2 = \cos y \)

Answer

The given relationship is \( ax + by^2 = \cos y \).

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} (ax) + \frac{d}{dx} (by^2) = \frac{d}{dx} (\cos y) \]

\[ \Rightarrow a + b \frac{d}{dx} (y^2) = \frac{d}{dx} (\cos y) \]

\[ ... (1) \]

Using chain rule, we obtain \( \frac{d}{dx} (y^2) = 2y \frac{dy}{dx} \) and \( \frac{d}{dx} (\cos y) = -\sin y \frac{dy}{dx} \) \( ... (2) \)

From (1) and (2), we obtain

\[ a + b \times 2y \frac{dy}{dx} = -\sin y \frac{dy}{dx} \]

\[ \Rightarrow (2by + \sin y) \frac{dy}{dx} = -a \]

\[ \therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y} \]

Question 4:

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\( xy + y^2 = \tan x + y \)
Answer

The given relationship is \( xy + y^2 = \tan x + y \)

Differentiating this relationship with respect to \( x \), we obtain
\[
\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)
\]
\[
\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}
\]
\[
\Rightarrow \left[ y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad \text{[Using product rule and chain rule]}
\]
\[
\Rightarrow y \cdot 1 + x \frac{dy}{dx} + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}
\]
\[
\Rightarrow (x + 2y - 1) \frac{dy}{dx} = \sec^2 x - y
\]
\[
\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}
\]

Question 5:

\[\frac{dy}{dx}\]

Find \( \frac{dx}{dy} \):

\[x^2 + xy + y^2 = 100\]

Answer

The given relationship is \( x^2 + xy + y^2 = 100 \)

Differentiating this relationship with respect to \( x \), we obtain
\[
\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)
\]
\[
\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0 \quad \text{[Derivative of constant function is 0]}
\]
Question 6:

\[ \frac{dy}{dx} \]
Find \( dx \):

\[ x^3 + x^2 y + xy^2 + y^3 = 81 \]

Answer

The given relationship is \( x^3 + x^2 y + xy^2 + y^3 = 81 \). Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} \left( x^3 + x^2 y + xy^2 + y^3 \right) = \frac{d}{dx} (81) \]

\[ \Rightarrow \frac{d}{dx} (x^3) + \frac{d}{dx} (x^2 y) + \frac{d}{dx} (xy^2) + \frac{d}{dx} (y^3) = 0 \]

\[ \Rightarrow 3x^2 + \left[ y \cdot \frac{d}{dx} (x^3) + x \cdot \frac{dy}{dx} \right] + \left[ y^2 \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (y^2) \right] + 3y^2 \frac{dy}{dx} = 0 \]

\[ \Rightarrow 3x^2 + \left[ y \cdot 2x + x \cdot \frac{dy}{dx} \right] + \left[ y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^2 \frac{dy}{dx} = 0 \]

\[ \Rightarrow \left( x^2 + 2xy + y^3 \right) \frac{dy}{dx} + (3x^2 + 2xy + y^3) = 0 \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{(3x^2 + 2xy + y^3)}{(x^2 + 2xy + y^3)} \]

Question 7:

\[ \frac{dy}{dx} \]
Find \( dx \):

\[ \sin^2 y + \cos xy = \pi \]
Answer

The given relationship is \( \sin^2 y + \cos xy = \pi \)

Differentiating this relationship with respect to \( x \), we obtain

\[
\frac{d}{dx} (\sin^2 y + \cos xy) = \frac{d}{dx} (\pi)
\]

\[
\Rightarrow \frac{d}{dx} (\sin^2 y) + \frac{d}{dx} (\cos xy) = 0 \quad \text{... (1)}
\]

Using chain rule, we obtain

\[
\frac{d}{dx} (\sin^2 y) = 2 \sin y \frac{d}{dx} (\sin y) = 2 \sin y \cos y \frac{dy}{dx} \quad \text{... (2)}
\]

\[
\frac{d}{dx} (\cos xy) = -\sin xy \frac{d}{dx} (xy) = -\sin xy \left[ y \frac{d}{dx} (x) + x \frac{dy}{dx} \right]
\]

\[
= -\sin xy \left[ y + x \frac{dy}{dx} \right] = -y \sin xy - x \sin xy \frac{dy}{dx} \quad \text{... (3)}
\]

From (1), (2), and (3), we obtain

\[
2 \sin y \cos y \frac{dy}{dx} - y \sin xy - x \sin xy \frac{dy}{dx} = 0
\]

\[
\Rightarrow (2 \sin y \cos y - x \sin xy) \frac{dy}{dx} = y \sin xy
\]

\[
\Rightarrow \left( \sin 2y - x \sin xy \right) \frac{dy}{dx} = y \sin xy
\]

\[
:\Rightarrow \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}
\]

Question 8:

\[
\frac{dy}{dx}
\]

Find \( \frac{dy}{dx} : \)

\( \sin^2 x + \cos^2 y = 1 \)

Answer

The given relationship is \( \sin^2 x + \cos^2 y = 1 \)

Differentiating this relationship with respect to \( x \), we obtain
Question 9:

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\[ y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \]

Answer

The given relationship is

\[ y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \]

\[ \Rightarrow \sin y = \frac{2x}{1+x^2} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} (\sin y) = \frac{d}{dx} \left(\frac{2x}{1+x^2}\right) \]

\[ \Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x}{1+x^2}\right) \quad \text{...(1)} \]

The function, \( \frac{2x}{1+x^2} \), is of the form \( \frac{u}{v} \).

Therefore, by quotient rule, we obtain
\[
\frac{d}{dx} \left( \frac{2x}{1+x^2} \right) = \frac{d}{dx} \left( \frac{2x}{1+x^2} \right) - 2x \cdot \frac{d}{dx} \left( \frac{1+x^2}{1+x^2} \right) \\
= \frac{(1+x^2) \cdot 2 - 2x \cdot [0 + 2x]}{(1+x^2)^2} = \frac{2 + 2x^2 - 4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2} \quad \ldots (2)
\]

\[
\sin y = \frac{2x}{1+x^2}
\]

Also,
\[
\Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left( \frac{2x}{1+x^2} \right)^2} = \sqrt{\frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}} = \frac{\sqrt{(1-x^2)^2}}{\sqrt{1+x^2}} = \frac{1-x^2}{1+x^2} \quad \ldots (3)
\]

From (1), (2), and (3), we obtain
\[
\frac{1-x^2}{1+x^2} \cdot \frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}
\]

**Question 10:**

\[
\frac{dy}{dx} = \tan^{-1} \left( \frac{3x - x^3}{1-3x^2} \right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}
\]

**Answer:**

\[
y = \tan^{-1} \left( \frac{3x - x^3}{1-3x^2} \right)
\]

The given relationship is
It is known that,
Comparing equations (1) and (2), we obtain

\[ x = \tan \frac{y}{3} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{dy}{dx} = \frac{3}{1 + x^2} \]

**Question 11:**

Find \( \frac{dy}{dx} \):

\[ y = \cos^{-1}\left( \frac{1 - x^2}{1 + x^2} \right), \quad 0 < x < 1 \]

**Answer**

The given relationship is,
\[ y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) \]
\[ \Rightarrow \cos y = \frac{1 - x^2}{1 + x^2} \]
\[ \Rightarrow \frac{1 - \tan^2 \frac{y}{2}}{2} = \frac{1 - x^2}{1 + x^2} \]

On comparing L.H.S. and R.H.S. of the above relationship, we obtain
\[ \tan \frac{y}{2} = x \]

Differentiating this relationship with respect to \( x \), we obtain
\[ \sec^2 \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x) \]
\[ \Rightarrow \sec^2 \frac{y}{2} \cdot \frac{1}{2} \frac{dy}{dx} = 1 \]
\[ \Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{2}{1 + \tan^2 \frac{y}{2}} \]
\[ \therefore \frac{dy}{dx} = \frac{1}{1 + x^2} \]

Question 12:
\[ \frac{dy}{dx} \]
Find \( dx \):
\[ y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), \ 0 < x < 1 \]

Answer
\[ y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) \]
The given relationship is
\[ y = \sin^{-1} \left( \frac{1-x^2}{1+x^2} \right) \]

\[ \Rightarrow \sin y = \frac{1-x^2}{1+x^2} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} (\sin y) = \frac{d}{dx} \left( \frac{1-x^2}{1+x^2} \right) \quad ... (1) \]

Using chain rule, we obtain

\[ \frac{d}{dx} (\sin y) = \cos y \cdot \frac{dy}{dx} \]

\[ \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left( \frac{1-x^2}{1+x^2} \right)^2} \]

\[ = \sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2} \]

\[ \therefore \frac{d}{dx} (\sin y) = \frac{2x}{1+x^2} \cdot \frac{dy}{dx} \quad ... (2) \]

\[ \frac{d}{dx} \left( \frac{1-x^2}{1+x^2} \right) = \frac{(1+x^2)' \cdot (1-x^2)' - (1-x^2) \cdot (1+x^2)'}{(1+x^2)^2} \quad \text{[Using quotient rule]} \]

\[ = \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2} \]

\[ = \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2} \]

\[ = \frac{-4x}{(1+x^2)^2} \quad ... (3) \]

From (1), (2), and (3), we obtain
Alternate method

\[ y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) \]

\[ \sin y = \frac{1-x^2}{1+x^2} \]

\[ \Rightarrow (1+x^2)\sin y = 1-x^2 \]
\[ \Rightarrow (1+\sin y)x^2 = 1-\sin y \]
\[ \Rightarrow x^2 = \frac{1-\sin y}{1+\sin y} \]

\[ \Rightarrow x = \frac{\cos \frac{y}{2} - \sin \frac{y}{2}}{\cos \frac{y}{2} + \sin \frac{y}{2}} \]
\[ \Rightarrow x = \frac{1 - \tan \frac{y}{2}}{1 + \tan \frac{y}{2}} \]
\[ \Rightarrow x = \tan \left( \frac{\pi - y}{4} \right) \]

Differentiating this relationship with respect to \( x \), we obtain
Question 13:

\[ \frac{dy}{dx} = \frac{-2}{1 + x^2} \]

Find \( \frac{dx}{dy} \):

\[ y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right), \quad -1 < x < 1 \]

Answer

The given relationship is

\[ y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right) \]

\[ \Rightarrow \cos y = \frac{2x}{1 + x^2} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx}(\cos y) = \frac{d}{dx}\left(\frac{2x}{1 + x^2}\right) \]

\[ \Rightarrow -\sin y \cdot \frac{dy}{dx} = \frac{(1 + x^2) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1 + x^2)}{(1 + x^2)^2} \]
Question 14:

Find \( \frac{dy}{dx} \):

\[
y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}
\]

Answer

\[ y = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \]

Differentiating this relationship with respect to \( x \), we obtain
Question 15:

\[ \frac{dy}{dx} \]

Find \( \frac{dy}{dx} \):

\[ y = \sec^{-1}\left( \frac{1}{2x^2 - 1} \right), \quad 0 < x < \frac{1}{\sqrt{2}} \]

Answer

The given relationship is

\[ y = \sec^{-1}\left( \frac{1}{2x^2 - 1} \right) \]
\[ \Rightarrow \sec y = \frac{1}{2x^2 - 1} \]
\[ \Rightarrow \cos y = 2x^2 - 1 \]
\[ \Rightarrow 2x^2 = 1 + \cos y \]
\[ \Rightarrow 2x^2 = 2\cos^2 \frac{y}{2} \]
\[ \Rightarrow x = \cos \frac{y}{2} \]

Differentiating this relationship with respect to \( x \), we obtain

\[ \frac{d}{dx} \left( x \right) = \frac{d}{dx} \left( \cos \frac{y}{2} \right) \]
\[ \Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx} \left( \frac{y}{2} \right) \]
\[ \Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} \cdot \frac{1}{\sqrt{1 - \cos^2 \frac{y}{2}}} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}} \]
Exercise 5.4

Question 1:
Differentiate the following w.r.t. $x$:

\[
\frac{e^x}{\sin x}
\]

Answer

Let \( y = \frac{e^x}{\sin x} \)

By using the quotient rule, we obtain

\[
\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}
\]

\[
= \frac{\sin x (e^x) - e^x \cdot (\cos x)}{\sin^2 x}
\]

\[
= \frac{e^x (\sin x - \cos x)}{\sin^2 x}, \quad x \neq n\pi, n \in \mathbb{Z}
\]

Question 2:
Differentiate the following w.r.t. $x$:

\[
e^{\sin^{-1} x}
\]

Answer

Let \( y = e^{\sin^{-1} x} \)

By using the chain rule, we obtain
Question 2:
Show that the function given by \( f(x) = e^{2x} \) is strictly increasing on \( \mathbb{R} \).

Answer
Let \( x_1 \) and \( x_2 \) be any two numbers in \( \mathbb{R} \).
Then, we have:
\[ x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2) \]
Hence, \( f \) is strictly increasing on \( \mathbb{R} \).

Question 3:
Differentiate the following w.r.t. \( x \):
\[ e^{x^2} \]

Answer
Let \( y = e^{x^2} \)
By using the chain rule, we obtain
\[
\frac{dy}{dx} = \frac{d}{dx} \left( e^{x^2} \right) = e^{x^2} \cdot \frac{d}{dx} \left( x^2 \right) = e^{x^2} \cdot 2x = 2xe^{x^2}
\]

Question 4:
Differentiate the following w.r.t. \( x \):
\[ \sin \left( \tan^{-1} e^{-x} \right) \]
Answer

Let \( y = \sin\left(\tan^{-1} e^{-x}\right) \)

By using the chain rule, we obtain

\[
\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right] \\
= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right) \\
= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1 + (e^{-x})^2} \cdot \frac{d}{dx}\left(e^{-x}\right) \\
= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1 + e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}\left(-x\right) \\
= \frac{e^{-x} \cos\left(\tan^{-1} e^{-x}\right)}{1 + e^{-2x}} \times (-1) \\
= -e^{-x} \cos\left(\tan^{-1} e^{-x}\right) \\
= -e^{-x} \cos\left(\tan^{-1} e^{-x}\right)
\]

Question 5:
Differentiate the following w.r.t. \( x \):

\( \log\left(\cos e^x\right) \)

Answer

Let \( y = \log\left(\cos e^x\right) \)

By using the chain rule, we obtain

\[
\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right] \\
= \frac{1}{\cos e^x} \cdot \frac{d}{dx}\left(\cos e^x\right) \\
= \frac{1}{\cos e^x} \cdot \left(-\sin e^x\right) \cdot \frac{d}{dx}\left(e^x\right) \\
= -\sin e^x \cdot e^x \cdot \frac{1}{\cos e^x} \\
= -e^x \tan e^x, \quad e^x \neq (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{N}
\]
Question 6:
Differentiate the following w.r.t. $x$:

$$e^x + e^{x^2} + ... + e^{x^n}$$

Answer

$$\frac{d}{dx}(e^x + e^{x^2} + ... + e^{x^n})$$

$$= \frac{d}{dx}(e^x) + \frac{d}{dx}(e^{x^2}) + \frac{d}{dx}(e^{x^3}) + \frac{d}{dx}(e^{x^4}) + \frac{d}{dx}(e^{x^n})$$

$$= e^x + \left[ e^{x^2} \cdot \frac{d}{dx}(x^2) \right] + \left[ e^{x^3} \cdot \frac{d}{dx}(x^3) \right] + \left[ e^{x^4} \cdot \frac{d}{dx}(x^4) \right] + \left[ e^{x^n} \cdot \frac{d}{dx}(x^n) \right]$$

$$= e^x + (e^{x^2} \cdot 2x) + (e^{x^3} \cdot 3x^2) + (e^{x^4} \cdot 4x^3) + (e^{x^n} \cdot n x^{n-1})$$

$$= e^x + 2xe^{x^2} + 3xe^{x^3} + 4xe^{x^4} + 5xe^{x^n}$$

Question 7:
Differentiate the following w.r.t. $x$:

$$\sqrt[3]{e^x}, x > 0$$

Answer

Let $y = \sqrt[3]{e^x}$

Then, $y^2 = e^x$

By differentiating this relationship with respect to $x$, we obtain
Question 8:
Differentiate the following w.r.t. $x$:

$\log(\log x), x > 1$

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx} \left[ \log(\log x) \right]$$
$$= \frac{1}{\log x} \cdot \frac{d}{dx} (\log x)$$
$$= \frac{1}{\log x} \cdot \frac{1}{x}$$
$$= \frac{1}{x\log x}, x > 1$$

Question 9:
Differentiate the following w.r.t. $x$:

$$\frac{\cos x}{\log x}, x > 0$$
Answer

Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\left(\cos x \times \log x - \cos x \times \frac{d}{dx}(\log x)\right)}{(\log x)^2}$$

$$= -\sin x \log x - \cos x \times \frac{1}{x}$$

$$= \frac{-\left[x \log x \sin x + \cos x\right]}{x(\log x)^2}, x > 0$$

**Question 10:**
Differentiate the following w.r.t. $x$:

$\cos (\log x + e^x), x > 0$

**Answer**

Let $y = \cos (\log x + e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin \left(\log x + e^x\right) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin \left(\log x + e^x\right) \cdot \left[\frac{1}{x} + e^x\right]$$

$$= -\left(\frac{1}{x} + e^x\right) \sin \left(\log x + e^x\right), x > 0$$
Exercise 5.5

Question 1:
Differentiate the function with respect to \( x \).

\[ \cos x \cos 2x \cos 3x \]

Answer
Let \( y = \cos x \cos 2x \cos 3x \)

Taking logarithm on both the sides, we obtain

\[ \log y = \log(\cos x \cos 2x \cos 3x) \]

\[ \Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x) \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \frac{d}{dx}(\cos x) + \frac{1}{\cos 2x} \frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x} \frac{d}{dx}(\cos 3x) \]

\[ \Rightarrow \frac{dy}{dx} = y \left[ -\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \cdot \frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x} \cdot \frac{d}{dx}(3x) \right] \]

\[ \therefore \frac{dy}{dx} = -\cos x \cos 2x \cos 3x \left[ \tan x + 2 \tan 2x + 3 \tan 3x \right] \]

Question 2:
Differentiate the function with respect to \( x \).

\[ \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \]

Answer
Let \( y = \sqrt[3]{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \)

Taking logarithm on both the sides, we obtain
\[ \log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \]

\[ \Rightarrow \log y = \frac{1}{2} \log \left[ \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right] \]

\[ \Rightarrow \log y = \frac{1}{2} \left[ \log \{(x-1)(x-2)\} - \log \{(x-3)(x-4)(x-5)\} \right] \]

\[ \Rightarrow \log y = \frac{1}{2} \left[ \log (x-1) + \log (x-2) - \log (x-3) - \log (x-4) - \log (x-5) \right] \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[ \frac{1}{x-1} \frac{d}{dx} (x-1) + \frac{1}{x-2} \frac{d}{dx} (x-2) - \frac{1}{x-3} \frac{d}{dx} (x-3) \right] \]

\[ \Rightarrow \frac{dy}{dx} = \frac{y}{2} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{2} \left[ \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right] \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \]

**Question 3:**

Differentiate the function with respect to \( x \).

\((\log x)^{\cos x}\)

**Answer**

Let \( y = (\log x)^{\cos x} \)

Taking logarithm on both the sides, we obtain

\[ \log y = \cos x \cdot \log (\log x) \]

Differentiating both sides with respect to \( x \), we obtain
Differentiate the function with respect to \( x \).

\( x^x - 2^{\sin x} \)

Answer

Let \( y = x^x - 2^{\sin x} \)

Also, let \( x^x = u \) and \( 2^{\sin x} = v \)

\[ y = u - v \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \]

\( u = x^x \)

Taking logarithm on both the sides, we obtain

\[ \log u = x \log x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{u} \cdot \frac{du}{dx} = \left[ \frac{d}{dx} (x) \times \log x + x \times \frac{d}{dx} (\log x) \right] \]

\[ \Rightarrow \frac{du}{dx} = u \left[ x \times \log x + x \times \frac{1}{x} \right] \]

\[ \Rightarrow \frac{du}{dx} = x^x (\log x + 1) \]

\[ v = 2^{\sin x} \]

Taking logarithm on both the sides with respect to \( x \), we obtain
\[ \log v = \sin x \cdot \log 2 \]

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx} (\sin x) \\
\Rightarrow \frac{dv}{dx} = v \log 2 \cos x \\
\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2 \\
\therefore \frac{dy}{dx} = x^2 (1 + \log x) - 2^{\sin x} \cos x \log 2
\]

**Question 5:**

Differentiate the function with respect to \( x \).

\[ (x + 3)^2 \cdot (x + 4)^2 \cdot (x + 5)^3 \]

**Answer**

Let \( y = (x + 3)^2 \cdot (x + 4)^2 \cdot (x + 5)^3 \)

Taking logarithm on both the sides, we obtain

\[ \log y = \log (x + 3)^2 + \log (x + 4)^2 + \log (x + 5)^3 \]

\[ \Rightarrow \log y = 2 \log (x + 3) + 3 \log (x + 4) + 4 \log (x + 5) \]

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x + 3} \cdot \frac{d}{dx} (x + 3)^2 + 3 \cdot \frac{1}{x + 4} \cdot \frac{d}{dx} (x + 4)^2 + 4 \cdot \frac{1}{x + 5} \cdot \frac{d}{dx} (x + 5)^3
\]

\[ \Rightarrow \frac{dy}{dx} = y \left[ \frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5} \right] \\
\Rightarrow \frac{dy}{dx} = (x + 3)^2 \cdot (x + 4)^2 \cdot (x + 5)^3 \cdot \left[ \frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5} \right] \\
\Rightarrow \frac{dy}{dx} = (x + 3)^2 \cdot (x + 4)^2 \cdot (x + 5)^3 \cdot \left[ \frac{2(x + 4)(x + 5) + 3(x + 3)(x + 5) + 4(x + 3)(x + 4)}{(x + 3)(x + 4)(x + 5)} \right] \\
\Rightarrow \frac{dy}{dx} = (x + 3)(x + 4)^2 \cdot (x + 5)^3 \cdot \left[ 2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12) \right] \\
\therefore \frac{dy}{dx} = (x + 3)(x + 4)^2 \cdot (x + 5)^3 \left( 9x^2 + 70x + 133 \right)
Question 6:
Differentiate the function with respect to $x$.

\[ \left( x + \frac{1}{x} \right)^x + x^{\left( \frac{x+1}{x} \right)} \]

Answer

Let $y = \left( x + \frac{1}{x} \right)^x + x^{\left( \frac{x+1}{x} \right)}$

Also, let $u = \left( x + \frac{1}{x} \right)^x$ and $v = x^{\left( \frac{x+1}{x} \right)}$

\[ \therefore y = u + v \]

\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...\(1\) \]

Then, $u = \left( x + \frac{1}{x} \right)^x$

\[ \Rightarrow \log u = \log \left( x + \frac{1}{x} \right)^x \]

\[ \Rightarrow \log u = x \log \left( x + \frac{1}{x} \right) \]

Differentiating both sides with respect to $x$, we obtain
\[ \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \left( \log \left( x + \frac{1}{x} \right) + x \log \left( x + \frac{1}{x} \right) \right) \]

\[ \Rightarrow \frac{1}{u} \frac{du}{dx} = 1 \times \log \left( x + \frac{1}{x} \right) + x \times \frac{1}{x + 1} \frac{d}{dx} \left( x + \frac{1}{x} \right) \]

\[ \Rightarrow \frac{du}{dx} = u \left[ \log \left( x + \frac{1}{x} \right) + \frac{x}{x + 1} \times \left( 1 - \frac{1}{x^2} \right) \right] \]

\[ \Rightarrow \frac{du}{dx} = \left( x + \frac{1}{x} \right) ^x \left[ \log \left( x + \frac{1}{x} \right) + \frac{x - 1}{x + 1} \right] \]

\[ \Rightarrow \frac{du}{dx} = \left( x + \frac{1}{x} \right) ^x \left[ \log \left( x + \frac{1}{x} \right) + \frac{x^2 - 1}{x^2 + 1} \right] \]

\[ \Rightarrow \frac{du}{dx} = \left( x + \frac{1}{x} \right) ^x \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left( x + \frac{1}{x} \right) \right] \]

\[ \Rightarrow 2 \; \log v = \log \left( \left( x + \frac{1}{x} \right) ^x \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left( x + \frac{1}{x} \right) \right] \right) \]

Differentiating both sides with respect to \( x \), we obtain

\[ \Rightarrow \log v = \left( 1 + \frac{1}{x} \right) \log x \]
\[
\frac{1}{v} \cdot \frac{dv}{dx} = \left[ \frac{d}{dx} \left( \frac{1 + \frac{1}{x}}{x} \right) \right] \times \log x + \left( 1 + \frac{1}{x} \right) \frac{d}{dx} \log x
\]

\[
\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \left(- \frac{1}{x^2}\right) \log x + \left( 1 + \frac{1}{x} \right) \frac{1}{x}
\]

\[
\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{-\log x + 1 + \frac{1}{x}}{x^2}
\]

\[
\Rightarrow \frac{dv}{dx} = v \left[ \frac{-\log x + x + 1}{x^2} \right]
\]

\[
\Rightarrow \frac{dv}{dx} = x^{\left(\frac{1}{x}\right)} \left( x + 1 - \frac{\log x}{x^2} \right)
\]

(3)

Therefore, from (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = \left(x + \frac{1}{x}\right) \left[ \frac{x^2 - 1}{x^2} + \log \left( x + \frac{1}{x} \right) \right] + x^{\left(\frac{1}{x}\right)} \left( x + 1 - \frac{\log x}{x^2} \right)
\]

Question 7:
Differentiate the function with respect to x.

\[
(\log x)^x + x^{\log x}
\]

Answer

Let \( y = (\log x)^x + x^{\log x} \)

Also, let \( u = (\log x)^x \) and \( v = x^{\log x} \)

\[
\Rightarrow y = u + v
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots (1)
\]

\( u = (\log x)^x \)

\[
\Rightarrow \log u = \log \left( (\log x)^x \right)
\]

\[
\Rightarrow \log u = x \log (\log x)
\]

Differentiating both sides with respect to x, we obtain
\[
\frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \left( x \log(x) + x \cdot \frac{d}{dx} \left( \log(x) \right) \right)
\]
\[
\Rightarrow \frac{du}{dx} = u \left[ 1 \cdot \log(x) + x \cdot \frac{1}{\log(x)} \cdot \frac{1}{x} \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log(x))^2 \left[ \log(x) + \frac{1}{\log(x)} \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log(x))^2 \left[ \frac{\log(x) \cdot \log(x) + 1}{\log(x)} \right]
\]
\[
\Rightarrow \frac{du}{dx} = (\log(x))^{x-1} \left[ 1 + \log(x) \cdot \log(\log(x)) \right] \tag{2}
\]

\(v = x^{\log x}\)

\[
\Rightarrow \log v = \log \left( x^{\log x} \right)
\]
\[
\Rightarrow \log v = \log x \cdot \log x = (\log x)^2
\]

Differentiating both sides with respect to \(x\), we obtain

\[
\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left[ (\log(x))^2 \right]
\]
\[
\Rightarrow \frac{1}{v} \frac{dv}{dx} = 2(\log(x)) \cdot \frac{d}{dx} (\log(x))
\]
\[
\Rightarrow \frac{dv}{dx} = 2v \cdot \frac{1}{x}
\]
\[
\Rightarrow \frac{dv}{dx} = 2x^{\log x} \cdot \frac{\log x}{x} \tag{3}
\]
\[
\Rightarrow \frac{dv}{dx} = 2x^{\log x-1} \cdot \log x
\]

Therefore, from (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = (\log(x))^{x-1} \left[ 1 + \log(x) \cdot \log(\log(x)) \right] + 2x^{\log x-1} \cdot \log x
\]

**Question 8:**

Differentiate the function with respect to \(x\).
\((\sin x)^r + \sin^{-1}\sqrt{x}\)

**Answer**

Let \(y = (\sin x)^r + \sin^{-1}\sqrt{x}\)

Also, let \(u = (\sin x)^r\) and \(v = \sin^{-1}\sqrt{x}\)

\[\therefore \, y = u + v\]

\[\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots (1)\]

\(u = (\sin x)^r\)

\[\Rightarrow \log u = \log (\sin x)^r\]

\[\Rightarrow \log u = r \log (\sin x)\]

Differentiating both sides with respect to \(x\), we obtain

\[\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (\log (\sin x)) + x \frac{d}{dx} \left[ \frac{r}{x} \log (\sin x) \right]\]

\[\Rightarrow \frac{du}{dx} = u \left[ \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right] + \frac{r}{\sin x} \cdot \frac{d}{dx} (\log (\sin x))\]

\[\Rightarrow \frac{du}{dx} = (\sin x)^r \left[ \log (\sin x) + \frac{1}{\sin x} \cdot \cos x \right] \quad \ldots (2)\]

\(v = \sin^{-1}\sqrt{x}\)

Differentiating both sides with respect to \(x\), we obtain

\[\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx} (\sqrt{x})\]

\[\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}\]

\[\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \quad \ldots (3)\]

Therefore, from (1), (2), and (3), we obtain

\[\frac{dy}{dx} = (\sin x)^r \left( x \cot x + \log \sin x \right) + \frac{1}{2\sqrt{x-x^2}}\]
Question 9:
Differentiate the function with respect to $x$.

$$x^{\sin x} + (\sin x)^{\cos x}$$

Answer

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(1)$$

$u = x^{\sin x}$

$$\Rightarrow \log u = \log(x^{\sin x})$$

$$\Rightarrow \log u = \sin x \log x$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \left(\sin x \cdot \log x + \sin x \cdot \frac{d}{dx} \left(\log x\right)\right)$$

$$\Rightarrow \frac{du}{dx} = u \left[ \cos x \log x + \sin x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x} \left[ \cos x \log x + \frac{\sin x}{x}\right] \quad \ldots(2)$$

$v = (\sin x)^{\cos x}$

$$\Rightarrow \log v = \log (\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log (\sin x)$$

Differentiating both sides with respect to $x$, we obtain
Differentiating both sides with respect to $x$, we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x} \left( \cos x \log x + \frac{\sin x}{x} \right) + (\sin x)^{\cos x} \left[ \cos x \cot x - \sin x \log \sin x \right]$$
\[
\frac{1}{u} \frac{du}{dx} = \frac{d}{dx} \left( x \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx} (\cos x) \cdot \log x + x \cdot \cos x \cdot \frac{d}{dx} (\log x) \right)
\]

\[
\Rightarrow \frac{du}{dx} = u \left[ -\cos x \cdot \log x + x \cdot (\sin x) \cdot \log x + x \cdot \cos x \cdot \frac{1}{x} \right]
\]

\[
\Rightarrow \frac{du}{dx} = x^\cos x \left( \cos x \cdot \log x - x \sin x \cdot \log x + \cos x \right)
\]

\[
\Rightarrow \frac{du}{dx} = x^\cos x \left[ \cos x (1 + \log x) - x \sin x \log x \right] \quad \text{...(2)}
\]

\[
v = \frac{x^2 + 1}{x^2 - 1}
\]

\[
\Rightarrow \log v = \log \left(\frac{x^2 + 1}{x^2 - 1}\right)
\]

Differentiating both sides with respect to \(x\), we obtain

\[
\frac{1}{v} \frac{dv}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}
\]

\[
\Rightarrow \frac{dv}{dx} = v \left[ \frac{2x (x^2 - 1) - 2x (x^2 + 1)}{(x^2 + 1)(x^2 - 1)} \right] \quad \text{...(3)}
\]

From (1), (2), and (3), we obtain

\[
\frac{dy}{dx} = x^\cos x \left[ \cos x (1 + \log x) - x \sin x \log x \right] - \frac{4x}{(x^2 - 1)^2}
\]

**Question 11:**
Differentiate the function with respect to \(x\).

\[
(x \cos x)^\frac{1}{x} + (x \sin x)^\frac{1}{x}
\]
Answer

Let \( y = (\cos x)^x + (\sin x)^x \)

Also, let \( u = (\cos x)^x \) and \( v = (\sin x)^x \)

\[ \therefore y = u + v \]
\[ \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \]

\[ u = (\cos x)^x \]
\[ \Rightarrow \log u = x \log (\cos x) \]
\[ \Rightarrow \log u = x \left[ \log x + \log \cos x \right] \]
\[ \Rightarrow \log u = x \log x + x \log \cos x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x \log x) + \frac{d}{dx} (x \log (\cos x)) \]
\[ \Rightarrow \frac{du}{dx} = u \left[ \log x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log x) \right] + \left[ \log \cos x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log (\cos x)) \right] \]
\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ \left( \log x + 1 + \frac{1}{x} \right) + \log \cos x + x \cdot \frac{d}{dx} (\log (\cos x)) \right] \]
\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ \left( \log x + 1 \right) + \left( \log \cos x + x \cdot \frac{d}{dx} (\log (\cos x)) \right) \right] \]
\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ 1 + \log x + \log \cos x - x \tan x \right] \]
\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ 1 - x \tan x + \log (\cos x) \right] \]
\[ \Rightarrow \frac{du}{dx} = (\cos x)^x \left[ 1 - x \tan x + \log (\cos x) \right] \]

\[ \Rightarrow \frac{dy}{dx} = (\cos x)^x \left[ 1 - x \tan x + \log (\cos x) \right] \]

\[ \cdots (2) \]
\[ v = \left( x \sin x \right)^{1} \]

\[ \Rightarrow \log v = \log \left( x \sin x \right)^{1} \]

\[ \Rightarrow \log v = \frac{1}{x} \log (x \sin x) \]

\[ \Rightarrow \log v = \frac{1}{x} \left( \log x + \log \sin x \right) \]

\[ \Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left[ \frac{1}{x} \log x \right] + \frac{d}{dx} \left[ \frac{1}{x} \log \sin x \right] \]

\[ \Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[ \log x \cdot \frac{d}{dx} \left( \frac{1}{x} \right) + \frac{1}{x} \frac{d}{dx} (\log x) \right] + \left[ \log \sin x \cdot \frac{d}{dx} \left( \frac{1}{x} \right) + \frac{1}{x} \frac{d}{dx} (\log \sin x) \right] \]

\[ \Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[ \log x \left( -\frac{1}{x^2} \right) + \frac{1}{x} \frac{1}{x} \right] + \left[ \log \sin x \left( -\frac{1}{x^2} \right) + \frac{1}{x} \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right] \]

\[ \Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2} \left( 1 - \log x \right) + \left[ \frac{\log \sin x}{x^2} - \frac{1}{x \sin x} \cdot \frac{d}{dx} (\sin x) \right] \]

\[ \Rightarrow \frac{dv}{dx} = \left( x \sin x \right) \left[ \frac{1 - \log x - \log \sin x + x \cot x}{x^2} \right] \]

\[ \Rightarrow \frac{dv}{dx} = \left( x \sin x \right) \left[ \frac{1 - \log (x \sin x) + x \cot x}{x^2} \right] \]

\[ \Rightarrow \frac{dv}{dx} = \left( x \sin x \right) \left[ \frac{1 - \log (x \sin x) + x \cot x}{x^2} \right] \quad \text{... (3)} \]

From (1), (2), and (3), we obtain

\[ \frac{dy}{dx} = \left( x \cos x \right) \left[ 1 - \tan x + \log (x \cos x) \right] + \left( x \sin x \right) \left[ \frac{x \cot x + 1 - \log (x \sin x)}{x^2} \right] \]

**Question 12:**

Find \( dx \) of function.
\[ x^y + y^x = 1 \]

Answer

The given function is \( x^y + y^x = 1 \)

Let \( x^y = u \) and \( y^x = v \)

Then, the function becomes \( u + v = 1 \)

\[ \therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \quad \ldots (1) \]

\( u = x^y \)

\[ \Rightarrow \log u = \log (x^y) \]

\[ \Rightarrow \log u = y \log x \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{u} \frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{1}{x} (\log x) \]

\[ \Rightarrow \frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{1}{x} \quad \ldots (2) \]

\( v = y^x \)

\[ \Rightarrow \log v = \log (y^x) \]

\[ \Rightarrow \log v = x \log y \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{v} \frac{dv}{dx} = \log y \cdot \frac{d}{dx} \left( \frac{x}{y} \right) + y \cdot \frac{1}{y} \cdot \frac{dy}{dx} \]

\[ \Rightarrow \frac{dv}{dx} = v \left( \log y \cdot \left( \frac{1}{x} - \frac{1}{y} \right) \right) \]

\[ \Rightarrow \frac{dv}{dx} = y^x \left( \log y + \frac{x}{y} \frac{dy}{dx} \right) \quad \ldots (3) \]

From (1), (2), and (3), we obtain
Question 13:
\[ \frac{dy}{dx} \]
Find \( \frac{dy}{dx} \) of function.
\[ y^x = x^x \]
Answer
The given function is \( y^x = x^x \)
Taking logarithm on both the sides, we obtain
\[ x \log y = y \log x \]
Differentiating both sides with respect to \( x \), we obtain
\[ \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x) \]
\[ \Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x} \]
\[ \Rightarrow \log y + \frac{x}{y} \cdot \frac{dy}{dx} = \frac{dy}{dx} \log x + \frac{y}{x} \]
\[ \Rightarrow \left( \frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y \]
\[ \Rightarrow \left( \frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{y}{x} \left( \frac{y - x \log y}{x - y \log x} \right) \]

Question 14:
\[ \frac{dy}{dx} \]
Find \( \frac{dy}{dx} \) of function.
\[(\cos x)^y = (\cos y)^x\]

Answer

The given function is \((\cos x)^y = (\cos y)^x\)

Taking logarithm on both the sides, we obtain

\[y \log \cos x = x \log \cos y\]

Differentiating both sides, we obtain

\[
\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx} \left(\log \cos x\right) = \log \cos y \cdot \frac{d}{dx} \left(x\right) + x \cdot \frac{d}{dx} \left(\log \cos y\right)
\]

\[
\Rightarrow \log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx} \left(\cos x\right) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx} \left(\cos y\right)
\]

\[
\Rightarrow \log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot (-\sin x) = \log \cos y + x \cdot \frac{1}{\cos y} \cdot \left(-\sin y\right) \cdot \frac{dy}{dx}
\]

\[
\Rightarrow \log \cos x \cdot \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \cdot \frac{dy}{dx}
\]

\[
\Rightarrow \left(\log \cos x + x \tan y\right) \cdot \frac{dy}{dx} = y \tan x + \log \cos y
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}
\]

**Question 15:**

Find \(\frac{dy}{dx}\) of function.

\[xy = e^{(x-y)}\]

Answer

The given function is \(xy = e^{(x-y)}\)

Taking logarithm on both the sides, we obtain

\[\log (xy) = \log \left(\frac{e^{x-y}}{x}\right)\]

\[
\Rightarrow \log x + \log y = (x - y) \log e
\]

\[
\Rightarrow \log x + \log y = (x - y) \times 1
\]

\[
\Rightarrow \log x + \log y = x - y
\]
Differentiating both sides with respect to \( x \), we obtain

\[
\frac{d}{dx} \left( \log x \right) + \frac{d}{dx} \left( \log y \right) = \frac{d}{dx} (x) - \frac{dy}{dx}
\]

\[
\Rightarrow \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 1 - \frac{dy}{dx}
\]

\[
\Rightarrow \left( 1 + \frac{1}{y} \right) \frac{dy}{dx} = 1 - \frac{1}{x}
\]

\[
\Rightarrow \left( \frac{y+1}{y} \right) \frac{dy}{dx} = \frac{x-1}{x}
\]

\[
\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}
\]

**Question 16:**

Find the derivative of the function given by \( f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8) \) and hence find \( f''(1) \).

**Answer**

The given relationship is \( f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8) \)

Taking logarithm on both sides, we obtain

\[
\log f(x) = \log (1+x) + \log (1+x^2) + \log (1+x^4) + \log (1+x^8)
\]

Differentiating both sides with respect to \( x \), we obtain
Question 17:

Differentiate \((x^5 - 5x + 8)(x^3 + 7x + 9)\) in three ways mentioned below

(i) By using product rule.
(ii) By expanding the product to obtain a single polynomial.
(iii) By logarithmic differentiation.

Do they all give the same answer?

Answer

Let \(y = (x^5 - 5x + 8)(x^3 + 7x + 9)\)

(i)
Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

\[
\therefore y = uv
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \quad \text{(By using product rule)}
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{d}{dx} (x^2 - 5x + 8) \cdot (x^3 + 7x + 9) + (x^2 - 5x + 8) \cdot \frac{d}{dx} (x^3 + 7x + 9)
\]

\[
\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)
\]

\[
\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7) - 5x(3x^2 + 7) + 8(3x^2 + 7)
\]

\[
\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56
\]

\[
\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11
\]

(ii)

\[
y = (x^2 - 5x + 8)(x^3 + 7x + 9)
\]

\[
= x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9)
\]

\[
= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^3 - 45x + 8x^3 + 56x + 72
\]

\[
= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72
\]

\[
\therefore \frac{dy}{dx} = \frac{d}{dx} (x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)
\]

\[
= \frac{d}{dx} (x^5) - 5 \frac{d}{dx} (x^4) + 15 \frac{d}{dx} (x^3) - 26 \frac{d}{dx} (x^2) + 11 \frac{d}{dx} (x) + \frac{d}{dx} (72)
\]

\[
= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0
\]

\[
= 5x^4 - 20x^3 + 45x^2 - 52x + 11
\]

(iii) \[
y = (x^2 - 5x + 8)(x^3 + 7x + 9)
\]

Taking logarithm on both the sides, we obtain

\[
\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)
\]

Differentiating both sides with respect to $x$, we obtain
\[
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \log(x^2 - 5x + 8) + \frac{d}{dx} \log(x^3 + 7x + 9)
\]
\[
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2 - 5x + 8} \frac{dx}{dx} (x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9} \frac{dx}{dx} (x^3 + 7x + 9)
\]
\[
\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{x^2 - 5x + 8} (2x - 5) + \frac{1}{x^3 + 7x + 9} (3x^2 + 7) \right]
\]
\[
\Rightarrow \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9) \left[ \frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9} \right]
\]
\[
\Rightarrow \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9) \left[ \frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right]
\]
\[
\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + 3x^2(x^2 - 5x + 8) + 7(x^2 - 5x + 8)
\]
\[
\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^3 + 18x) - 5x^3 - 35x - 45 + (3x^2 - 15x^3 + 24x^2) + (7x^2 - 35x + 56)
\]
\[
\Rightarrow \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11
\]

From the above three observations, it can be concluded that all the results of \( \frac{dy}{dx} \) are same.

Question 18:
If \( u, v \) and \( w \) are functions of \( x \), then show that
\[
\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + \frac{dv}{dx}w.u + \frac{dw}{dx}u.v
\]
in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer
\[
y = u.v.w = u.(v.w)
\]
Let

By applying product rule, we obtain
\[
\frac{dy}{dx} = \frac{du}{dx} \cdot (v \cdot w) + u \cdot \frac{d}{dx} (v \cdot w) \\
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx} \quad \text{(Again applying product rule)} \\
\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx} \\
\]

By taking logarithm on both sides of the equation \(y = u \cdot v \cdot w\), we obtain
\[\log y = \log u + \log v + \log w\]

Differentiating both sides with respect to \(x\), we obtain
\[
\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} (\log u) + \frac{d}{dx} (\log v) + \frac{d}{dx} (\log w) \\
\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} \\
\Rightarrow \frac{dy}{dx} = y \left( \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} \right) \\
\Rightarrow \frac{dy}{dx} = u \cdot v \cdot w \left( \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} \right) \\
\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx} \]
Exercise 5.6

Question 1:
If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2, \ y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

Then, $\frac{dx}{dt} = 2a \cdot t^3 = 2at^2$

$\frac{dy}{dt} = a \cdot 4t^3 = 4at^3$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{2at^2} = \frac{2t}{1}$

Question 2:
If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \cos \theta, \ y = b \cos \theta$

Answer

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

Then, $\frac{dx}{d\theta} = a \cdot (-\sin \theta) = -a \sin \theta$

$\frac{dy}{d\theta} = b \cdot (-\sin \theta) = -b \sin \theta$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b \sin \theta}{a \sin \theta} = \frac{b}{a}$
Question 3:
If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \sin t, \ y = \cos 2t$

Answer

The given equations are $x = \sin t$ and $y = \cos 2t$.

Then,

\[
\frac{dx}{dt} = \frac{d}{dt} (\sin t) = \cos t
\]

\[
\frac{dy}{dt} = \frac{d}{dt} (\cos 2t) = -\sin 2t \cdot \frac{d}{dt} (2t) = -2\sin 2t
\]

\[
\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t
\]

Question 4:
If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 4t, \ y = \frac{4}{t}$

Answer

The given equations are

$x = 4t$ and $y = \frac{4}{t}$.
Question 5:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\[
x = \cos \theta - \cos 2\theta, \quad y = \sin \theta - \sin 2\theta
\]

Answer

The given equations are \( x = \cos \theta - \cos 2\theta \) and \( y = \sin \theta - \sin 2\theta \)

Then,

\[
\frac{dx}{d\theta} = \frac{d}{d\theta} (\cos \theta - \cos 2\theta) = -\sin \theta - 2\sin 2\theta
\]

\[
\frac{dy}{d\theta} = \frac{d}{d\theta} (\sin \theta - \sin 2\theta) = \cos \theta - 2\cos 2\theta
\]

\[
\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta - 2\cos 2\theta}{2\sin 2\theta - \sin \theta}
\]

Question 6:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\[
x = a(\theta - \sin \theta), \quad y = a(1 + \cos \theta)
\]
Answer

The given equations are

\[ x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 + \cos \theta) \]

Then,

\[
\frac{dx}{d\theta} = a \left[ \frac{d}{d\theta} (\theta) - \frac{d}{d\theta} (\sin \theta) \right] = a(1 - \cos \theta)
\]

\[
\frac{dy}{d\theta} = a \left[ \frac{d}{d\theta} (1) + \frac{d}{d\theta} (\cos \theta) \right] = a[0 + (\cos \theta)] = a \sin \theta
\]

\[
\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-\sin \theta}{2 \sin^2 \frac{\theta}{2}} = \frac{-\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} = -\cot \frac{\theta}{2}
\]

**Question 7:**

If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\[ x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, \quad y = \frac{\cos^3 t}{\sqrt{\cos 2t}} \]

Answer

The given equations are

\[ x = \frac{\sin^3 t}{\sqrt{\cos 2t}} \quad \text{and} \quad y = \frac{\cos^3 t}{\sqrt{\cos 2t}} \]
\[
\frac{dx}{dt} = \frac{d}{dt} \left( \frac{\sin^3 t}{\cos 2t} \right) \\
= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt} (\sin^3 t) - \sin^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t} \\
= \frac{\sqrt{\cos 2t} \cdot 3 \sin^2 t \cdot \frac{d}{dt} (\sin t) - \sin^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t} \\
= \frac{3 \sqrt{\cos 2t} \cdot \sin^3 t \cos t - \sin^3 t \cdot \frac{1}{2 \sqrt{\cos 2t}} \cdot \frac{d}{dt} (\cos 2t)}{\cos 2t} \\
= \frac{3 \cos 2t \sin^3 t \cos t + \sin^3 t \sin 2t}{\cos 2t \sqrt{\cos 2t}}
\]

\[
\frac{dy}{dt} = \frac{d}{dt} \left( \frac{\cos^3 t}{\cos 2t} \right) \\
= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt} (\cos^3 t) - \cos^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t} \\
= \frac{\sqrt{\cos 2t} \cdot 3 \cos^2 t \cdot \frac{d}{dt} (\cos t) - \cos^3 t \cdot \frac{1}{2 \sqrt{\cos 2t}} \cdot \frac{d}{dt} (\cos 2t)}{\cos 2t} \\
= \frac{3 \sqrt{\cos 2t} \cdot \cos^2 t \cdot (-\sin t) - \cos^3 t \cdot \frac{1}{2 \sqrt{\cos 2t}} \cdot (-2 \sin 2t)}{\cos 2t} \\
= \frac{-3 \cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t \cdot \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}
\]
Question 8:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).

\[ x = a \left( \cos t + \log \tan \frac{t}{2} \right), \quad y = a \sin t \]

Answer

\[ x = a \left( \cos t + \log \tan \frac{t}{2} \right) \] and \( y = a \sin t \)

The given equations are
Then, \( \frac{dx}{dt} = a \cdot \left[ \frac{d}{dt} \left( \cos t \right) + \frac{d}{dt} \left( \log \tan \frac{t}{2} \right) \right] \)

\[ = a \left[ -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt} \left( \tan \frac{t}{2} \right) \right] \]

\[ = a \left[ -\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt} \left( \frac{t}{2} \right) \right] \]

\[ = a \left[ -\sin t + \frac{\cos \frac{t}{2}}{2} \times \frac{1}{\sin \frac{t}{2}} \times \frac{1}{\cos \frac{t}{2}} \right] \]

\[ = a \left[ -\sin t + \frac{1}{2 \sin^2 \frac{t}{2}} \right] \]

\[ = a \left( -\sin t + \frac{1}{\sin t} \right) \]

\[ = a \left( -\sin^2 t + 1 \right) \]

\[ = a \frac{\cos^2 t}{\sin t} \]

\[ \frac{dy}{dt} = a \frac{d}{dt} \left( \sin t \right) = a \cos t \]

\[ \therefore \frac{dy}{dx} = \left( \frac{dy}{dt} \right) \cdot \left( \frac{dt}{dx} \right) = \frac{a \cos t}{\cos^2 t} \cdot \frac{\sin t}{\tan t} = \tan t \]

**Question 9:**

If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( dx \).

\( x = a \sec \theta \), \( y = b \tan \theta \)
Answer

The given equations are \( x = a \sec \theta \) and \( y = b \tan \theta \)

Then, \[
\frac{dx}{d\theta} = a \cdot \frac{d}{d\theta} (\sec \theta) = a \sec \theta \tan \theta
\]
\[
\frac{dy}{d\theta} = b \cdot \frac{d}{d\theta} (\tan \theta) = b \sec^2 \theta
\]
\[
\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b}{a} \sec \theta \cot \theta = \frac{b}{a} \frac{\cos \theta}{\sin \theta} = \frac{b}{a} \frac{1}{\sin \theta} = \frac{b}{a} \csc \theta
\]

Question 10:
If \( x \) and \( y \) are connected parametrically by the equation, without eliminating the parameter, find \( \frac{dy}{dx} \).
\[x = a \left( \cos \theta + \theta \sin \theta \right), \quad y = a \left( \sin \theta - \theta \cos \theta \right)\]

Answer

The given equations are \( x = a \left( \cos \theta + \theta \sin \theta \right) \) and \( y = a \left( \sin \theta - \theta \cos \theta \right) \)

Then, \[
\frac{dx}{d\theta} = a \left[ \frac{d}{d\theta} \cos \theta + \frac{d}{d\theta} (\theta \sin \theta) \right] = a \left[ -\sin \theta + \theta \cdot \frac{d}{d\theta} (\sin \theta) + \sin \theta \cdot \frac{d}{d\theta} (\theta) \right]
\]
\[
= a \left[ -\sin \theta + \theta \cos \theta + \sin \theta \right] = a \theta \cos \theta
\]
\[
\frac{dy}{d\theta} = a \left[ \frac{d}{d\theta} (\sin \theta) - \frac{d}{d\theta} (\theta \cos \theta) \right] = a \left[ \cos \theta - \left( \theta \cdot \frac{d}{d\theta} (\cos \theta) + \cos \theta \cdot \frac{d}{d\theta} (\theta) \right) \right]
\]
\[
= a \left[ \cos \theta + \theta \sin \theta - \cos \theta \right]
\]
\[
= a \theta \sin \theta
\]
\[
\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta
\]
Question 11:

If \( x = \sqrt{a^{\sin^{-1} t}}, \ y = \sqrt{a^{\cos^{-1} t}} \), show that \( \frac{dy}{dx} = -\frac{y}{x} \)

Answer

The given equations are \( x = \sqrt{a^{\sin^{-1} t}} \) and \( y = \sqrt{a^{\cos^{-1} t}} \)

\[ x = \sqrt{a^{\sin^{-1} t}} \quad \text{and} \quad y = \sqrt{a^{\cos^{-1} t}} \]

\[ \Rightarrow x = \left(a^{\sin^{-1} t}\right)^{\frac{1}{2}} \quad \text{and} \quad y = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}} \]

\[ \Rightarrow x = a^{\frac{1}{2}\sin^{-1} t} \quad \text{and} \quad y = a^{\frac{1}{2}\cos^{-1} t} \]

Consider \( x = a^{\frac{1}{2}\sin^{-1} t} \)

Taking logarithm on both the sides, we obtain

\[ \log x = \frac{1}{2} \sin^{-1} t \log a \]

\[ \therefore \frac{1}{x} \frac{dx}{dt} = \frac{1}{2} \log a \cdot \frac{d}{dt} (\sin^{-1} t) \]

\[ \Rightarrow \frac{dx}{dt} = \frac{x}{2} \frac{\log a}{\sqrt{1-t^2}} \]

\[ \Rightarrow \frac{dx}{dt} = \frac{x \log a}{2 \sqrt{1-t^2}} \]

Then, consider \( y = a^{\frac{1}{2}\cos^{-1} t} \)

Taking logarithm on both the sides, we obtain

\[ \log y = \frac{1}{2} \cos^{-1} t \log a \]

\[ \therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \log a \cdot \frac{d}{dt} (\cos^{-1} t) \]

\[ \Rightarrow \frac{dy}{dt} = \frac{y}{2} \log a \left( \frac{-1}{\sqrt{1-t^2}} \right) \]

\[ \Rightarrow \frac{dy}{dt} = -\frac{y \log a}{2 \sqrt{1-t^2}} \]
\[
\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(-\frac{y \log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x \log a}{2\sqrt{1-t^2}}\right)} = \frac{y}{x}.
\]

Hence, proved.
Exercise 5.7

**Question 1:**
Find the second order derivatives of the function.

\[ x^2 + 3x + 2 \]

Answer

Let \( y = x^2 + 3x + 2 \)

Then,

\[
\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2
\]

**Question 2:**
Find the second order derivatives of the function.

\[ x^{20} \]

Answer

Let \( y = x^{20} \)

Then,

\[
\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 20 \frac{d}{dx}(x^{19}) = 20 \cdot 19 \cdot x^{18} = 380x^{18}
\]

**Question 3:**
Find the second order derivatives of the function.

\[ x \cdot \cos x \]

Answer

Let \( y = x \cdot \cos x \)

Then,
\[
\frac{dy}{dx} = \frac{d}{dx} (x \cdot \cos x) = \cos x \cdot \frac{d}{dx} (x) + x \frac{d}{dx} (\cos x) = \cos x \cdot 1 + x (-\sin x) = \cos x - x \sin x
\]

\[
\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ \cos x - x \sin x \right] = \frac{d}{dx} (\cos x) - \frac{d}{dx} (x \sin x)
\]

\[
= -\sin x - \left[ \sin x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\sin x) \right]
\]

\[
= -\sin x - (\sin x + x \cos x)
\]

\[
= -(x \cos x + 2 \sin x)
\]

**Question 4:**
Find the second order derivatives of the function.

\[\log x\]

**Answer**

Let \( y = \log x \)

Then,

\[
\frac{dy}{dx} = \frac{d}{dx} (\log x) = \frac{1}{x}
\]

\[
\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}
\]

**Question 5:**
Find the second order derivatives of the function.

\[x^3 \log x\]

**Answer**

Let \( y = x^3 \log x \)

Then,
\[
\frac{dy}{dx} = \frac{d}{dx} \left[ x^3 \log x \right] = \log x \cdot \frac{d}{dx} \left( x^3 \right) + x^3 \cdot \frac{d}{dx} \left( \log x \right) \\
= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2 \\
= x^3 \left( 1 + 3 \log x \right)
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ x^3 \left( 1 + 3 \log x \right) \right] \\
= \left( 1 + 3 \log x \right) \cdot \frac{d}{dx} \left( x^3 \right) + x^3 \frac{d}{dx} \left( 1 + 3 \log x \right) \\
= \left( 1 + 3 \log x \right) \cdot 2x + x^2 \cdot \frac{3}{x} \\
= 2x + 6x \log x + 3x \\
= 5x + 6x \log x \\
= x \left( 5 + 6 \log x \right)
\]

**Question 6:**
Find the second order derivatives of the function.

\[e^x \sin 5x\]

**Answer**

Let \(y = e^x \sin 5x\)

\[
\frac{dy}{dx} = e^x \sin 5x \cdot \frac{d}{dx} \left( e^x \right) + e^x \frac{d}{dx} \left( \sin 5x \right) \\
= e^x (5 \cos 5x) + e^x \sin 5x \\
= e^x \left( 5 \cos 5x + \sin 5x \right)
\]

\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ e^x \left( 5 \cos 5x + \sin 5x \right) \right] \\
= \left( 5 \cos 5x + \sin 5x \right) \cdot \frac{d}{dx} \left( e^x \right) + e^x \frac{d}{dx} \left( 5 \cos 5x + \sin 5x \right) \\
= \left( 5 \cos 5x + \sin 5x \right) e^x + e^x \left[ \cos 5x \cdot \frac{d}{dx} (5x) + 5 \left( - \sin 5x \right) \cdot \frac{d}{dx} (5x) \right] \\
= e^x \left( 5 \cos 5x + \sin 5x \right) + 5 e^x \cos 5x - 25 e^x \sin 5x \\
= e^x (5 \cos 5x + 10 \sin 5x - 24 \sin 5x) = 2 e^x (5 \cos 5x - 12 \sin 5x)
\]
Question 7:
Find the second order derivatives of the function.
\[ e^{6x} \cos 3x \]
Answer
Let \( y = e^{6x} \cos 3x \)
Then,
\[
\frac{dy}{dx} = e^{6x} \cdot \frac{d}{dx} (e^{6x} \cos 3x) = e^{6x} \cdot (e^{6x} \cdot 6x + \cos 3x \cdot 3x) \\
= 6e^{6x} \cos 3x + 3e^{6x} \sin 3x \cdot 6x \\
\]
\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} (6e^{6x} \cos 3x - 3e^{6x} \sin 3x) = 6 \cdot \frac{d}{dx} (e^{6x} \cos 3x) - 3 \cdot \frac{d}{dx} (e^{6x} \sin 3x) \\
= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3 \left[ \sin 3x \cdot e^{6x} \cdot 6x + e^{6x} \cdot 6x \cdot 3x \right] \\
= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x + 9e^{6x} \cos 3x \\
= 27e^{6x} \cos 3x - 54e^{6x} \sin 3x \\
= 9e^{6x} (3 \cos 3x - 4 \sin 3x) \quad \text{(1)}
\]

Question 8:
Find the second order derivatives of the function.
\[ \tan^{-1} x \]
Answer
Let \( y = \tan^{-1} x \)
Then,
Question 9:
Find the second order derivatives of the function.

\[ \log(\log x) \]

Answer

Let \( y = \log(\log x) \)

Then,

\[ \frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) = \frac{1}{x \log x} \]

\[ \Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{(x \log x)^2} \left[ \log x \cdot \frac{d}{dx} x + x \cdot \frac{d}{dx} (\log x) \right] \]

\[ = \frac{1}{(x \log x)^2} \left[ \log x \cdot 1 + x \cdot \frac{1}{x} \right] = \frac{(1 + \log x)}{(x \log x)^3} \]

Question 10:
Find the second order derivatives of the function.

\[ \sin(\log x) \]

Answer

Let \( y = \sin(\log x) \)

Then,
Question 11:

If \( y = 5 \cos x - 3 \sin x \), prove that \( \frac{d^2 y}{dx^2} + y = 0 \).

Answer

It is given that, \( y = 5 \cos x - 3 \sin x \).

Then,
\[
\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5 \frac{d}{dx}(\cos x) - 3 \frac{d}{dx}(\sin x)
\]
\[
= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)
\]
\[
\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[-(5\sin x + 3\cos x)]
\]
\[
= -\left[5\frac{d}{dx}(\sin x) + 3\frac{d}{dx}(\cos x)\right]
\]
\[
= -[5\cos x + 3(-\sin x)]
\]
\[
= -[5\cos x - 3\sin x]
\]
\[
= -y
\]
\[
\therefore \frac{d^2y}{dx^2} + y = 0
\]

Hence, proved.

**Question 12:**

If \( y = \cos^{-1} x \), find \( \frac{d^2y}{dx^2} \) in terms of \( y \) alone.

**Answer**

It is given that, \( y = \cos^{-1} x \)

Then,
\[
\frac{dy}{dx} = \frac{d}{dx} \left( \cos^{-1} x \right) = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}}
\]
\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ -(1-x^2)^{-\frac{1}{2}} \right]
\]
\[
= -\left( -\frac{1}{2} \right) (1-x^2)^{-\frac{3}{2}} \cdot \frac{d}{dx} (1-x^2)
\]
\[
= \frac{1}{2\sqrt{(1-x^2)^3}} \times (-2x)
\]
\[
\Rightarrow \frac{d^2 y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}} \tag{i}
\]

Let \( y = \cos^{-1} x \) \( \Rightarrow x = \cos y \)

Putting \( x = \cos y \) in equation (i), we obtain
\[
\frac{d^2 y}{dx^2} = \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}}
\]
\[
\Rightarrow \frac{d^2 y}{dx^2} = \frac{-\cos y}{\sqrt{(\sin^2 y)^3}}
\]
\[
= -\frac{\cos y}{\sin y \cdot \sin^3 y}
\]
\[
= -\frac{\cos y}{\sin^3 y}
\]
\[
\Rightarrow \frac{d^3 y}{dx^3} = -\cot y \cdot \cosec^3 y
\]

**Question 13:**

If \( y = 3\cos (\log x) + 4\sin (\log x) \), show that \( x^2 y'' + xy' + y = 0 \)

**Answer**

It is given that, \( y = 3\cos (\log x) + 4\sin (\log x) \)

Then,
\[ y_1 = 3 \frac{d}{dx} \left[ \cos (\log x) \right] + 4 \frac{d}{dx} \left[ \sin (\log x) \right] \]
\[ = 3 \left[ -\sin (\log x) \frac{d}{dx} (\log x) \right] + 4 \left[ \cos (\log x) \frac{d}{dx} (\log x) \right] \]
\[ \therefore y_1 = \frac{-3 \sin (\log x)}{x} + \frac{4 \cos (\log x)}{x} = \frac{4 \cos (\log x) - 3 \sin (\log x)}{x} \]

\[ y_2 = \frac{d}{dx} \left( \frac{4 \cos (\log x) - 3 \sin (\log x)}{x} \right) \]
\[ = \frac{x \left\{ 4 \cos (\log x) - 3 \sin (\log x) \right\}' - \left\{ 4 \cos (\log x) - 3 \sin (\log x) \right\} (x)'}{x^2} \]
\[ = \frac{x \left\{ 4 \left( \cos (\log x) \right)' - 3 \left( \sin (\log x) \right)' \right\} - \left\{ 4 \cos (\log x) - 3 \sin (\log x) \right\} (1)}{x^2} \]
\[ = \frac{x \left\{ -4 \sin (\log x) \left( \log x \right)' - 3 \cos (\log x) \left( \log x \right)' \right\} - 4 \cos (\log x) + 3 \sin (\log x)}{x^2} \]
\[ = \frac{x \left\{ -4 \sin (\log x) \cdot \frac{1}{x} - 3 \cos (\log x) \cdot \frac{1}{x} \right\} - 4 \cos (\log x) + 3 \sin (\log x)}{x^2} \]
\[ = \frac{-4 \sin (\log x) - 3 \cos (\log x) - 4 \cos (\log x) + 3 \sin (\log x)}{x^2} \]
\[ = \frac{-\sin (\log x) - 7 \cos (\log x)}{x^2} \]

\[ \therefore x^2 y_2 + xy_1 + y = \frac{- \sin (\log x) - 7 \cos (\log x)}{x^2} + \left( \frac{4 \cos (\log x) - 3 \sin (\log x)}{x} \right) + 3 \cos (\log x) + 4 \sin (\log x) \]
\[ = - \sin (\log x) - 7 \cos (\log x) + 4 \cos (\log x) - 3 \sin (\log x) + 3 \cos (\log x) + 4 \sin (\log x) \]
\[ = 0 \]

Hence, proved.

**Question 14:**

If \[ y = Ae^{mx} + Be^{nx} \], show that \[ \frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0 \]
Answer

It is given that, \( y = Ae^{mx} + Be^{nx} \)

Then,

\[
\frac{dy}{dx} = A \cdot \frac{d}{dx}(e^{mx}) + B \cdot \frac{d}{dx}(e^{nx}) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am e^{mx} + Bn e^{nx}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx}\left( Am e^{mx} + Bn e^{nx} \right) = Am \cdot \frac{d}{dx}(e^{mx}) + Bn \cdot \frac{d}{dx}(e^{nx})
\]

\[
= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2 e^{mx} + Bn^2 e^{nx}
\]

\[
\therefore \frac{d^2y}{dx^2} = \left( m + n \right) \frac{dy}{dx} + mny
\]

\[
= Am^2 e^{mx} + Bn^2 e^{nx} - \left( m + n \right) \left( Am e^{mx} + Bn e^{nx} \right) + mn \left( Ae^{mx} + Be^{nx} \right)
\]

\[
= Am^2 e^{mx} + Bn^2 e^{nx} - Am e^{mx} - Bn e^{nx} - Am e^{mx} - Bn e^{nx} + Am e^{mx} + Bn e^{nx}
\]

\[
= 0
\]

Hence, proved.

Question 15:

If \( y = 500e^{7x} + 600e^{-7x} \), show that \( \frac{d^2y}{dx^2} = 49y \)

Answer

It is given that, \( y = 500e^{7x} + 600e^{-7x} \)

Then,
\[
\frac{dy}{dx} = 500 \cdot \frac{d}{dx} (e^{7x}) + 600 \cdot \frac{d}{dx} (e^{-7x})
\]
\[
= 500 \cdot e^{7x} \cdot \frac{d}{dx} (7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx} (-7x)
\]
\[
= 3500e^{7x} - 4200e^{-7x}
\]
\[
\therefore \frac{d^2 y}{dx^2} = 3500 \cdot \frac{d}{dx} (e^{7x}) - 4200 \cdot \frac{d}{dx} (e^{-7x})
\]
\[
= 3500 \cdot e^{7x} \cdot \frac{d}{dx} (7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx} (-7x)
\]
\[
= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}
\]
\[
= 49 \times 500e^{7x} + 49 \times 600e^{-7x}
\]
\[
= 49 \left( 500e^{7x} + 600e^{-7x} \right)
\]
\[
= 49y
\]
Hence, proved.

Question 16:

If \(e^{y} (x+1) = 1\), show that \(\frac{d^2 y}{dx^2} = \left( \frac{dy}{dx} \right)^2\)

Answer

The given relationship is
\(e^{y} (x+1) = 1\)
\(e^{y} (x+1) = 1\)

\(\Rightarrow e^{y} = \frac{1}{x+1}\)

Taking logarithm on both the sides, we obtain

\(y = \log \frac{1}{(x+1)}\)

Differentiating this relationship with respect to \(x\), we obtain
\[ \frac{dy}{dx} = (x+1) \frac{d}{dx} \left( \frac{1}{x+1} \right) = (x+1) \cdot \frac{-1}{(x+1)^2} = \frac{-1}{x+1} \]

\[ \therefore \frac{d^2y}{dx^2} = - \frac{d}{dx} \left( \frac{1}{x+1} \right) = -\left( \frac{-1}{(x+1)^2} \right) = \frac{1}{x+1} \]

\[ \Rightarrow \frac{d^2y}{dx^2} = \left( \frac{-1}{x+1} \right)^2 \]

\[ \Rightarrow \frac{d^2y}{dx^2} = \left( \frac{dy}{dx} \right)^2 \]

Hence, proved.

**Question 17:**

If \( y = \left( \tan^{-1} x \right)^2 \), show that \( \left( x^2 + 1 \right)^2 y_2 + 2x \left( x^2 + 1 \right) y_1 = 2 \)

**Answer**

The given relationship is \( y = \left( \tan^{-1} x \right)^2 \).

Then,

\[ y_1 = 2 \tan^{-1} x \frac{d}{dx} \left( \tan^{-1} x \right) \]

\[ \Rightarrow y_1 = 2 \tan^{-1} x \cdot \frac{1}{1+x^2} \]

\[ \Rightarrow (1+x^2) y_1 = 2 \tan^{-1} x \]

Again differentiating with respect to \( x \) on both the sides, we obtain

\[ (1+x^2) y_2 + 2x y_1 = 2 \left( \frac{1}{1+x^2} \right) \]

\[ \Rightarrow (1+x^2)^2 y_2 + 2x (1+x^2) y_1 = 2 \]

Hence, proved.
Question 1:

Verify Rolle’s Theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$

Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in $[-4, 2]$ and is differentiable in $(-4, 2)$.

\[ f(-4) = (-4)^2 + 2(-4) - 8 = 16 - 8 - 8 = 0 \]

\[ f(2) = (2)^2 + 2\times2 - 8 = 4 + 4 - 8 = 0 \]

\[ \therefore f(-4) = f(2) = 0 \]

⇒ The value of $f(x)$ at $-4$ and $2$ coincides.

Rolle’s Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

\[ f(x) = x^2 + 2x - 8 \]

\[ \Rightarrow f'(x) = 2x + 2 \]

\[ \therefore f'(c) = 0 \]

\[ \Rightarrow 2c + 2 = 0 \]

\[ \Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2) \]

Hence, Rolle’s Theorem is verified for the given function.
Question 2:
Examine if Rolle’s Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle’s Theorem from these examples?

(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

(ii) \( f(x) = [x] \) for \( x \in [-2, 2] \)

(iii) \( f(x) = x^2 - 1 \) for \( x \in [1, 2] \)

Answer

By Rolle’s Theorem, for a function \( f:[a, b] \to \mathbb{R} \), if

(a) \( f \) is continuous on \( [a, b] \)
(b) \( f \) is differentiable on \( (a, b) \)
(c) \( f(a) = f(b) \)

then, there exists some \( c \in (a, b) \) such that \( f'(c) = 0 \)

Therefore, Rolle’s Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point. In particular, \( f(x) \) is not continuous at \( x = 5 \) and \( x = 9 \)

\[ \Rightarrow f(x) \text{ is not continuous in } [5, 9]. \]

Also, \( f(5) = [5] = 5 \) and \( f(9) = [9] = 9 \)

\[ \therefore f(5) \neq f(9) \]

The differentiability of \( f \) in \( (5, 9) \) is checked as follows.
Let $n$ be an integer such that $n \in (5, 9)$.

The left hand limit of $f$ at $x = n$ is,

$$\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} 0 = 0$$

Since the left and right hand limits of $f$ at $x = n$ are not equal, $f$ is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(5, 9)$.

It is observed that $f$ does not satisfy all the conditions of the hypothesis of Rolle’s Theorem.

Hence, Rolle’s Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(iii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of $f$ in $(-2, 2)$ is checked as follows.
Let $n$ be an integer such that $n \in (-2, 2)$.

The left hand limit of $f$ at $x = n$ is,
\[
\lim_{h \to 0} \frac{f(n + h) - f(n)}{h} = \lim_{h \to 0} \frac{[n + h] - [n]}{h} = \lim_{h \to 0} \frac{n - 1 - n}{h} = \lim_{h \to 0} -1 = \infty
\]
The right hand limit of $f$ at $x = n$ is,
\[
\lim_{h \to 0} \frac{f(n + h) - f(n)}{h} = \lim_{h \to 0} \frac{[n + h] - [n]}{h} = \lim_{h \to 0} \frac{n - n}{h} = \lim_{h \to 0} 0 = 0
\]
Since the left and right hand limits of $f$ at $x = n$ are not equal, $f$ is not differentiable at $x = n$.

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that $f$ does not satisfy all the conditions of the hypothesis of Rolle’s Theorem.

Hence, Rolle’s Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^3 - 1$ for $x \in [1, 2]$

It is evident that $f$, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

\[
f(1) = (1)^3 - 1 = 0
\]
\[
f(2) = (2)^3 - 1 = 3
\]

$\therefore f(1) \neq f(2)$

It is observed that $f$ does not satisfy a condition of the hypothesis of Rolle’s Theorem.
Hence, Rolle’s Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

**Question 3:**

If $f : [-5, 5] \rightarrow \mathbb{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

**Answer**

It is given that $f : [-5, 5] \rightarrow \mathbb{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) $f$ is continuous on $[-5, 5]$.

(b) $f$ is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$\therefore f'(c) \neq 0$

$\Rightarrow 10f'(c) \neq 0$

$\Rightarrow f(5) - f(-5) \neq 0$

$\Rightarrow f(5) \neq f(-5)$

Hence, proved.

**Question 4:**

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$. 
The given function is \( f(x) = x^3 - 5x^2 - 3x \)

\( f \), being a polynomial function, is continuous in \([1, 4]\) and is differentiable in \((1, 4)\) whose derivative is \( 3x^2 - 10x - 3 \).

\[
\begin{align*}
  f(1) &= 1^3 - 5 \times 1^2 - 3 \times 1 = -7, \\
  f(4) &= 4^3 - 5 \times 4^2 - 3 \times 4 = -27 \\
  \therefore \quad \frac{f(4) - f(1)}{4 - 1} &= \frac{-27 - (-7)}{3} = \frac{10}{3}
\end{align*}
\]

Mean Value Theorem states that there is a point \( c \in (1, 4) \) such that \( f'(c) = 1 \)

\[
f'(c) = 1 \\
\Rightarrow \quad 2c - 4 = 1 \\
\Rightarrow \quad c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)
\]

Hence, Mean Value Theorem is verified for the given function.

**Question 5:**
Verify Mean Value Theorem, if \( f(x) = x^3 - 5x^2 - 3x \) in the interval \([a, b]\), where \( a = 1 \) and \( b = 3 \). Find all \( c \in (1, 3) \) for which \( f'(c) = 0 \)

**Answer**

The given function \( f \) is \( f(x) = x^3 - 5x^2 - 3x \)

\( f \), being a polynomial function, is continuous in \([1, 3]\) and is differentiable in \((1, 3)\) whose derivative is \( 3x^2 - 10x - 3 \).

\[
\begin{align*}
  f(1) &= 1^3 - 5 \times 1^2 - 3 \times 1 = -7, \\
  f(3) &= 3^3 - 5 \times 3^2 - 3 \times 3 = -27 \\
  \therefore \quad \frac{f(3) - f(1)}{3 - 1} &= \frac{-27 - (-7)}{3 - 1} = \frac{-10}{2} = -5
\end{align*}
\]
Mean Value Theorem states that there exist a point \( c \in (1, 3) \) such that \( f'(c) = -10 \)

\[
f'(c) = -10 \\
\Rightarrow 3c^2 - 10c - 3 = 10 \\
\Rightarrow 3c^2 - 10c + 7 = 0 \\
\Rightarrow 3c^2 - 3c - 7c + 7 = 0 \\
\Rightarrow 3c(c - 1) - 7(c - 1) = 0 \\
\Rightarrow (c - 1)(3c - 7) = 0 \\
\Rightarrow c = 1, \quad \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3)
\]

Hence, Mean Value Theorem is verified for the given function and \( c = \frac{7}{3} \in (1, 3) \) is the only point for which \( f'(c) = 0 \)

**Question 6:**
Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

**Answer**

Mean Value Theorem states that for a function \( f: [a, b] \rightarrow \mathbb{R} \), if

(a) \( f \) is continuous on \([a, b]\)
(b) \( f \) is differentiable on \((a, b)\)

then, there exists some \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.
(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point. In particular, \( f(x) \) is not continuous at \( x = 5 \) and \( x = 9 \)

\[ f(x) \text{ is not continuous in } [5, 9]. \]

The differentiability of \( f \) in \((5, 9)\) is checked as follows.

Let \( n \) be an integer such that \( n \in (5, 9) \).

The left hand limit of \( f \) at \( x = n \) is,

\[ \lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h] - [n]}{h} = \lim_{h \to 0} \frac{n-1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty \]

The right hand limit of \( f \) at \( x = n \) is,

\[ \lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h] - [n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} 0 = 0 \]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)

\[ \therefore f \text{ is not differentiable in } (5, 9). \]

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for \( f(x) = [x] \) for \( x \in [5, 9] \).

(ii) \( f(x) = [x] \) for \( x \in [-2, 2] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.
In particular, \( f(x) \) is not continuous at \( x = -2 \) and \( x = 2 \).

\[ \Rightarrow f(x) \text{ is not continuous in } [-2, 2]. \]

The differentiability of \( f \) in \((-2, 2)\) is checked as follows.

Let \( n \) be an integer such that \( n \in (-2, 2) \).

The left hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty
\]
The right hand limit of \( f \) at \( x = n \) is,
\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} 0 = 0
\]
Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)
\[ \therefore \text{ } f \text{ is not differentiable in } (-2, 2). \]

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for \( f(x) = \lceil x \rceil \) for \( x \in [-2, 2] \).

\( \text{(iii) } f(x) = x^2 - 1 \text{ for } x \in [1, 2] \)
It is evident that \( f \), being a polynomial function, is continuous in \([1, 2]\) and is differentiable in \((1, 2)\).
It is observed that \( f \) satisfies all the conditions of the hypothesis of Mean Value Theorem.
Hence, Mean Value Theorem is applicable for \( f(x) = x^2 - 1 \) for \( x \in [1, 2] \).

It can be proved as follows.

\[
\begin{align*}
f(1) &= 1^2 - 1 = 0, \quad f'(2) = 2^1 - 1 = 3 \\
\therefore \quad \frac{f(b) - f(a)}{b - a} &= \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3 \\
f'(x) &= 2x \\
\therefore \quad f'(c) &= 3 \\
\Rightarrow \quad 2c &= 3 \\
\Rightarrow \quad c &= \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]
\end{align*}
\]
Question 1:
\[(3x^2 - 9x + 5)^9\]

Answer

Let \(y = (3x^2 - 9x + 5)^9\)

Using chain rule, we obtain

\[
\frac{dy}{dx} = \frac{d}{dx}(3x^2 - 9x + 5)^9 \\
= 9(3x^2 - 9x + 5)^8 \cdot \frac{d}{dx}(3x^2 - 9x + 5) \\
= 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) \\
= 9(3x^2 - 9x + 5)^8 \cdot 3(2x - 3) \\
= 27(3x^2 - 9x + 5)^8 (2x - 3)
\]

Question 2:
\(\sin^3 x + \cos^6 x\)

Answer

Let \(y = \sin^3 x + \cos^6 x\)

\[
\frac{dy}{dx} = \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\cos^6 x) \\
= 3\sin^2 x \cdot \frac{d}{dx}(\sin x) + 6\cos^5 x \cdot \frac{d}{dx}(\cos x) \\
= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x) \\
= 3\sin x \cos x (\sin x - 2\cos^4 x)
\]

Question 3:
\[(5x)^{3\cos 2x}\]

Answer
Let \( y = (5x)^{3 \cos 2x} \)

Taking logarithm on both the sides, we obtain

\[
\log y = 3 \cos 2x \log 5x
\]

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{y} \frac{dy}{dx} = 3 \left[ \log 5x \cdot \frac{d}{dx} (\cos 2x) + \cos 2x \cdot \frac{d}{dx} (\log 5x) \right]
\]

\[
\Rightarrow \frac{dy}{dx} = 3y \left[ \log 5x (-\sin 2x) \cdot \frac{d}{dx} (2x) + \cos 2x \cdot \frac{1}{5x} \cdot \frac{d}{dx} (5x) \right]
\]

\[
\Rightarrow \frac{dy}{dx} = 3y \left[ -2 \sin 2x \log 5x + \cos 2x \cdot \frac{1}{5x} \cdot 5 \right]
\]

\[
\Rightarrow \frac{dy}{dx} = 3y \left[ \frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right]
\]

\[
\therefore \frac{dy}{dx} = (5x)^{3 \cos 2x} \left[ \frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right]
\]

**Question 4:**

\[\sin^{-1} \left( x \sqrt{x} \right), \ 0 \leq x \leq 1\]

**Answer**

Let \( y = \sin^{-1} \left( x \sqrt{x} \right) \)

Using chain rule, we obtain
\[
\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)
= \frac{1}{\sqrt{1-(x\sqrt{x})^2}} \times \frac{d}{dx}(x\sqrt{x})
= \frac{1}{\sqrt{1-x^3}} \times \frac{d}{dx}\left(x^\frac{3}{2}\right)
= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^\frac{1}{2}
= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}
= \frac{3}{2} \frac{x}{\sqrt{x^3+1}}
\]

Question 5:
\[
\cos^{-1}\frac{x}{2\sqrt{2x+7}}, -2 < x < 2
\]

Answer
Let \( y = \frac{\cos^{-1} x}{\sqrt{2x+7}} \)

By quotient rule, we obtain

\[
\frac{dy}{dx} = \frac{\sqrt{2x+7} \frac{d}{dx} \left( \frac{\cos^{-1} x}{2} \right) - \left( \frac{\cos^{-1} x}{2} \right) \frac{d}{dx} \left( \sqrt{2x+7} \right)}{(\sqrt{2x+7})^2}
\]

\[
= \frac{\sqrt{2x+7} \left[ \frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx} \left( \frac{x}{2} \right) \right] - \left( \frac{\cos^{-1} x}{2} \right) \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx} (2x+7)}{(\sqrt{2x+7})^2}
\]

\[
= \frac{\sqrt{2x+7} \frac{-1}{\sqrt{4-x^2}} \cdot \frac{x}{2}}{2x+7} - \frac{\left( \frac{\cos^{-1} x}{2} \right) \frac{2}{2\sqrt{2x+7}}}{2x+7}
\]

\[
= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2} \cdot (2x+7)} \cdot \frac{\cos^{-1} x}{2} - \frac{1}{\sqrt{4-x^2} \cdot \sqrt{2x+7} \cdot (2x+7)^2}
\]

\[
= \frac{1}{\sqrt{4-x^2} \cdot \sqrt{2x+7} \cdot (2x+7)^2}
\]

Question 6:

\[\cot^{-1} \left[ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}\]

Answer
Let \( y = \cot^{-1} \left[ \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right] \)  

(1)

Then, \[ \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} = \left( \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right)^{2} \]

\[ = \frac{(1 + \sin x) + (1 - \sin x) + 2\sqrt{(1 - \sin x)(1 + \sin x)}}{(1 + \sin x) - (1 - \sin x)} \]

\[ = \frac{2 + 2\sqrt{1 - \sin^2 x}}{2\sin x} \]

\[ = \frac{1 + \cos x}{\sin x} \]

\[ = \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \]

\[ = \cot \frac{x}{2} \]

Therefore, equation (1) becomes

\[ y = \cot^{-1} \left( \cot \frac{x}{2} \right) \]

\[ \Rightarrow y = \frac{x}{2} \]

\[ \therefore \frac{dy}{dx} = \frac{1}{2} \frac{d}{dx} (x) \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{2} \]

**Question 7:**

\((\log x)^{\log x}, \ x > 1\)

Answer

Let \( y = (\log x)^{\log x} \)
Taking logarithm on both the sides, we obtain
\[ \log y = \log x \cdot \log \left( \log x \right) \]

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left[ \log x \cdot \log \left( \log x \right) \right]
\]
\[
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\log x) \cdot \frac{d}{dx} \left( \log x \right) + \log x \cdot \frac{d}{dx} \left[ \log (\log x) \right]
\]
\[
\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{x} \log(\log x) + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) \right]
\]
\[
\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{x} \log(\log x) + 1 \right]
\]
\[
\Rightarrow \frac{dy}{dx} = (\log x)^{\log x} \left[ \frac{1}{x} + \frac{\log(\log x)}{x} \right]
\]

**Question 8:**
\[ \cos \left( a \cos x + b \sin x \right) \], for some constant \( a \) and \( b \).

**Answer**

Let \( y = \cos \left( a \cos x + b \sin x \right) \)

By using chain rule, we obtain
\[
\frac{dy}{dx} = \frac{d}{dx} \cos \left( a \cos x + b \sin x \right)
\]
\[
\Rightarrow \frac{dy}{dx} = -\sin \left( a \cos x + b \sin x \right) \cdot \frac{d}{dx} \left( a \cos x + b \sin x \right)
\]
\[
= -\sin \left( a \cos x + b \sin x \right) \cdot \left[ a (-\sin x) + b \cos x \right]
\]
\[
= (a \sin x - b \cos x) \cdot \sin \left( a \cos x + b \sin x \right)
\]

**Question 9:**
\[ \left( \sin x - \cos x \right)^{\sin x - \cos x} \], \( \frac{\pi}{4} < x < \frac{3\pi}{4} \)

**Answer**
Let \( y = (\sin x - \cos x)^{(\sin x - \cos x)} \)

Taking logarithm on both the sides, we obtain

\[
\log y = \log[(\sin x - \cos x)^{(\sin x - \cos x)}] \\
\Rightarrow \log y = (\sin x - \cos x) \cdot \log(\sin x - \cos x)
\]

Differentiating both sides with respect to \( x \), we obtain

\[
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx}[(\sin x - \cos x) \log(\sin x - \cos x)] \\
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\sin x - \cos x) \cdot \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \cdot \frac{d}{dx} \log(\sin x - \cos x) \\
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \frac{d}{dx}(\sin x - \cos x) \\
\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \left[ (\cos x + \sin x) \cdot \log(\sin x - \cos x) + (\cos x + \sin x) \right] \\
\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \cdot (\cos x + \sin x) \left[ 1 + \log(\sin x - \cos x) \right]
\]

**Question 10:**

\( x^x + x^a + a^x + a^a \), for some fixed \( a > 0 \) and \( x > 0 \)

**Answer**

Let \( y = x^x + x^a + a^x + a^a \).

Also, let \( x^x = u \), \( x^a = v \), \( a^x = w \), and \( a^a = s \)

\( \therefore y = u + v + w + s \)

\[
\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \quad \ldots(1)
\]

\( u = x^x \)

\( \Rightarrow \log u = \log x^x \)

\( \Rightarrow \log u = x \log x \)

Differentiating both sides with respect to \( x \), we obtain
\[ \frac{1}{u} \frac{du}{dx} = \log x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log x) \]

\[ \Rightarrow \frac{du}{dx} = u \left[ \log x \cdot 1 + x \cdot \frac{1}{x} \right] \]

\[ \Rightarrow \frac{du}{dx} = x^x \left[ \log x + 1 \right] = x^x (1 + \log x) \quad \ldots (2) \]

\( v = x^x \)

\[ \Rightarrow \frac{dv}{dx} = \frac{d}{dx} (x^x) \]

\[ \therefore \frac{dv}{dx} = ax^{a-1} \quad \ldots (3) \]

\( w = a^x \)

\[ \Rightarrow \log w = \log a^x \]

\[ \Rightarrow \log w = x \log a \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{1}{w} \frac{dw}{dx} = \log a \cdot \frac{d}{dx} (x) \]

\[ \Rightarrow \frac{dw}{dx} = w \log a \]

\[ \Rightarrow \frac{dw}{dx} = a^x \log a \quad \ldots (4) \]

\( s = a^x \)

Since \( a \) is constant, \( a^x \) is also a constant.

\[ \frac{ds}{dx} = 0 \quad \ldots (5) \]

\[ \therefore \frac{dy}{dx} = x^x (1 + \log x) + ax^{a-1} + a^x \log a + 0 \]

\[ = x^x (1 + \log x) + ax^{a-1} + a^x \log a \]

From (1), (2), (3), (4), and (5), we obtain
Question 11:

\[ x^{x^2-3} + (x-3)^x, \text{ for } x > 3 \]

Answer

Let \( y = x^{x^2-3} + (x-3)^x \)

Also, let \( u = x^{x^2-3} \) and \( v = (x-3)^x \)

\[ y = u + v \]

Differentiating both sides with respect to \( x \), we obtain

\[ \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \] \( \ldots (1) \)

\[ u = x^{x^2-3} \]

\[ \therefore \log u = \log(x^{x^2-3}) \]

\[ \log u = (x^2 - 3) \log x \]

Differentiating with respect to \( x \), we obtain

\[ \frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{d}{dx}(x^2-3) + (x^2-3) \cdot \frac{d}{dx}(\log x) \]

\[ \Rightarrow \frac{1}{u} \frac{du}{dx} = \log x \cdot 2x + (x^2-3) \cdot \frac{1}{x} \]

\[ \Rightarrow \frac{du}{dx} = x^2 + \left[ \frac{x^2 - 3}{x} + 2 \log x \right] \]

Also,

\[ v = (x-3)^x \]

\[ \therefore \log v = \log (x-3)^x \]

\[ \Rightarrow \log v = x \log (x-3) \]

Differentiating both sides with respect to \( x \), we obtain
\[
\frac{1}{v} \frac{dv}{dx} = \log(x-3) \cdot \frac{d}{dx} \left( x^2 \right) + x^2 \cdot \frac{d}{dx} \left[ \log(x-3) \right]
\]
\[
\Rightarrow \frac{1}{v} \frac{dv}{dx} = \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \frac{d}{dx} (x-3)
\]
\[
\Rightarrow \frac{dv}{dx} = v \left[ 2x \log(x-3) + \frac{x^2}{x-3} \cdot 1 \right]
\]
\[
\Rightarrow \frac{dv}{dx} = (x-3)^{\frac{1}{v}} \left[ \frac{x^2}{x-3} + 2x \log(x-3) \right]
\]

Substituting the expressions of \( \frac{dv}{dx} \) and \( \frac{dv}{dt} \) in equation (1), we obtain

\[
\frac{dy}{dx} = x^{x-3} \left[ \frac{x^2}{x-3} + 2x \log(x-3) \right] + (x-3)^{\frac{1}{v}} \left[ \frac{x^2}{x-3} + 2x \log(x-3) \right]
\]

**Question 12:**

\[
\frac{dy}{dx} = 12(1-\cos t), x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}
\]

**Find \( \frac{dx}{dt} \), if**

**Answer**

It is given that, \( y = 12(1-\cos t), x = 10(t - \sin t) \)

\[
\Rightarrow \frac{dx}{dt} = 10 \cdot \frac{d}{dt} (t - \sin t) = 10(1 - \cos t)
\]

\[
\frac{dy}{dt} = 12 \cdot \frac{d}{dt} (1 - \cos t) = 12 \cdot \left[ 0 - (-\sin t) \right] = 12 \sin t
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1 - \cos t)} = \frac{12 \cdot 2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}}{10 \cdot 2 \sin^2 \frac{t}{2}} = \frac{6 \cot \frac{t}{2}}{5}
\]

**Question 13:**

\[
\frac{dy}{dx} = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}, -1 \leq x \leq 1
\]

**Find \( \frac{dx}{dt} \), if**

**Answer**
It is given that, \( y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2} \)

\[
\frac{dy}{dx} = \frac{d}{dx} \left[ \sin^{-1} x + \sin^{-1} \sqrt{1-x^2} \right]
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left( \sin^{-1} x \right) + \frac{d}{dx} \left( \sin^{-1} \sqrt{1-x^2} \right)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx} \left( \sqrt{1-x^2} \right)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} \left( 1-x^2 \right)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}
\]

\[
\therefore \frac{dy}{dx} = 0
\]

**Question 14:**

If \( x\sqrt{1+y} + y\sqrt{1+x} = 0 \), for, \(-1 < x < 1\), prove that

\[
\frac{dy}{dx} = -\frac{1}{(1+x)^2}
\]

**Answer**

It is given that,

\( x\sqrt{1+y} + y\sqrt{1+x} = 0 \)
Differentiating both sides with respect to $x$, we obtain

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2 (1+y) = y^2 (1+x)$$

$$\Rightarrow x^2 + x^2 y = y^2 + xy^2$$

$$\Rightarrow x^2 - y^2 = xy^2 - x^2 y$$

$$\Rightarrow x^2 - y^2 = xy(y-x)$$

$$\Rightarrow (x+y)(x-y) = xy(y-x)$$

$$\therefore x+y = -xy$$

$$\Rightarrow (1+x)y = -x$$

$$\Rightarrow y = \frac{-x}{1+x}$$

Differentiating both sides with respect to $x$, we obtain

$$y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x) \frac{d}{dx} (x) - x \frac{d}{dx} (1+x)}{(1+x)^2} = \frac{(1+x) - x}{(1+x)^2} \frac{1}{(1+x)^2}$$

Hence, proved.

**Question 15:**

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] \frac{d^2y}{dx^2}$$

is a constant independent of $a$ and $b$.

**Answer**

It is given that $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to $x$, we obtain
\[
\frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] = \frac{d}{dx}(c^2)
\]

\[\Rightarrow 2(x-a) \cdot \frac{d}{dx}(x-a) + 2(y-b) \cdot \frac{d}{dx}(y-b) = 0\]

\[\Rightarrow 2(x-a) \cdot 1 + 2(y-b) \cdot \frac{dy}{dx} = 0\]

\[\Rightarrow \frac{dy}{dx} = \frac{(x-a)}{y-b} \quad \text{...}(1)\]

\[\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{(x-a)}{y-b}\right]\]
Hence, proved.

Question 16:

If \( \cos y = x \cos(a + y) \), with \( \cos a \neq \pm 1 \), prove that \( \frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a} \)
It is given that, \( \cos y = x \cos (a + y) \)
\[
\therefore \frac{dy}{dx} \left[ \cos y \right] = \frac{dy}{dx} \left[ x \cos (a + y) \right]
\]
\[\Rightarrow -\sin y \frac{dy}{dx} = \cos (a + y) \cdot \frac{dy}{dx} (x) + x \cdot \frac{dy}{dx} \left[ \cos (a + y) \right]
\]
\[\Rightarrow -\sin y \frac{dy}{dx} = \cos (a + y) + x \cdot \left[ -\sin (a + y) \right] \frac{dy}{dx}
\]
\[\Rightarrow \left[ x \sin (a + y) + \sin y \right] \frac{dy}{dx} = \cos (a + y) \quad \text{...(1)}
\]
Since \( \cos y = x \cos (a + y) \), \( x = \frac{\cos y}{\cos (a + y)} \)

Then, equation (1) reduces to
\[
\left[ \frac{\cos y}{\cos (a + y)} \cdot \sin (a + y) - \sin y \right] \frac{dy}{dx} = \cos (a + y)
\]
\[\Rightarrow \left[ \cos y \cdot \sin (a + y) - \sin y \cdot \cos (a + y) \right] \frac{dy}{dx} = \cos^2 (a + y)
\]
\[\Rightarrow \sin (a + y - y) \frac{dy}{dx} = \cos^2 (a + b)
\]
\[\Rightarrow \frac{dy}{dx} = \frac{\cos^2 (a + b)}{\sin a}
\]
Hence, proved.

**Question 17:**

If \( x = a (\cos t + t \sin t) \) and \( y = a (\sin t - t \cos t) \), find \( \frac{d^2 y}{dx^2} \)

**Answer**
It is given that, \( x = a(\cos t + t \sin t) \) and \( y = a(\sin t - t \cos t) \)

\[
\frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t \sin t) \\
= a \left[ -\sin t + \sin t \cdot \frac{d}{dx}(t) + t \cdot \frac{d}{dt}(\sin t) \right] \\
= a \left[ -\sin t + \sin t + t \cos t \right] = at \cos t
\]

\[
\frac{dy}{dt} = a \cdot \frac{d}{dt}(\sin t - t \cos t) \\
= a \left[ \cos t - \left\{ \cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t) \right\} \right] \\
= a \left[ \cos t - \{\cos t - t \sin t\} \right] = at \sin t
\]

\[
\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t} = \tan t
\]

Then, \[ \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} (\tan t) = \sec^2 t \cdot \frac{dt}{dx} \]

\[
= \sec^2 t \cdot \frac{1}{at \cos t} \Rightarrow \frac{dx}{dt} = at \cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at \cos t}
\]

\[
= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}
\]

**Question 18:**

If \( f(x) = |x|^3 \), show that \( f''(x) \) exists for all real \( x \), and find it.

**Answer**

\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0
\end{cases}
\]

It is known that,

Therefore, when \( x \geq 0 \), \( f(x) = |x|^3 = x^3 \)

In this case, \( f'(x) = 3x^2 \) and hence, \( f''(x) = 6x \)

When \( x < 0 \), \( f(x) = |x|^3 = (-x)^3 = -x^3 \)
In this case, \( f'(x) = -3x^2 \) and hence, \( f''(x) = -6x \)

Thus, for \( f(x) = |x|^3 \), \( f''(x) \) exists for all real \( x \) and is given by,
\[
    f''(x) = \begin{cases} 
        6x, & \text{if } x \geq 0 \\
        -6x, & \text{if } x < 0 
    \end{cases}
\]

Question 19:

Using mathematical induction prove that \( \frac{d}{dx}(x^n) = nx^{n-1} \) for all positive integers \( n \).

Answer

To prove: \( P(n) : \frac{d}{dx}(x^n) = nx^{n-1} \) for all positive integers \( n \)

For \( n = 1 \),
\[
P(1) : \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}
\]

\( \therefore P(n) \) is true for \( n = 1 \)

Let \( P(k) \) is true for some positive integer \( k \).

\[
P(k) : \frac{d}{dx}(x^k) = kx^{k-1}
\]

That is,

It has to be proved that \( P(k + 1) \) is also true.

Consider \( \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) \)

\[
= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \quad \text{[By applying product rule]}
\]

\[
= x^k \cdot 1 + x \cdot k \cdot x^{k-1}
\]

\[
= x^k + kx^k
\]

\[
= (k+1) \cdot x^k
\]

\[
= (k+1) \cdot x^{k-1+1-1}
\]
Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer $n$.

Hence, proved.

**Question 20:**
Using the fact that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

**Answer**

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{d}{dx} \left[ \sin(A + B) \right] = \frac{d}{dx} \left( \sin A \cos B + \cos A \sin B \right)$$

$$\Rightarrow \cos(A + B) \cdot \frac{d}{dx} (A + B) = \cos B \cdot \frac{d}{dx} (\sin A) + \sin A \cdot \frac{d}{dx} (\cos B)$$

$$+ \sin B \cdot \frac{d}{dx} (\cos A) + \cos A \cdot \frac{d}{dx} (\sin B)$$

$$\Rightarrow \cos(A + B) \cdot \frac{d}{dx} (A + B) = \cos B \cdot \cos A \cdot \frac{dA}{dx} + \sin A(-\sin B) \cdot \frac{dB}{dx}$$

$$+ \sin B(-\sin A) \cdot \frac{dA}{dx} + \cos A \cdot \cos B \frac{dB}{dx}$$

$$\Rightarrow \cos(A + B) \left[ \frac{dA}{dx} + \frac{dB}{dx} \right] = (\cos A \cos B - \sin A \sin B) \cdot \left[ \frac{dA}{dx} + \frac{dB}{dx} \right]$$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B$$

**Question 22:**

$$y = \begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

$$\frac{dy}{dx} = \begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

If $$y = \begin{bmatrix} \frac{dy}{dx} \end{bmatrix}$$

**Answer**
\[
\begin{align*}
\begin{vmatrix}
  f(x) & g(x) & h(x) \\
  l & m & n \\
  a & b & c
\end{vmatrix}
\Rightarrow y &= (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x) \\
\text{Then,} \quad \frac{dy}{dx} &= \frac{d}{dx}[(mc - nb)f(x)] - \frac{d}{dx}[(lc - na)g(x)] + \frac{d}{dx}[(lb - ma)h(x)] \\
&= (mc - nb)f'(x) - (lc - na)g'(x) + (lb - ma)h'(x) \\
&= \begin{vmatrix}
  f'(x) & g'(x) & h'(x) \\
  l & m & n \\
  a & b & c
\end{vmatrix}
\end{align*}
\]

Thus,

**Question 23:**

If \( y = e^{\cos x}, -1 \leq x \leq 1 \), show that \( \left(1 - x^2\right) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2y = 0 \)

**Answer**

It is given that \( y = e^{\cos x} \).
Taking logarithm on both the sides, we obtain
\[ \log y = a \cos^{-1} x \log e \]
\[ \log y = a \cos^{-1} x \]
Differentiating both sides with respect to \( x \), we obtain
\[ \frac{1}{y} \frac{dy}{dx} = a \frac{1}{\sqrt{1-x^2}} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \]
By squaring both the sides, we obtain
\[ \left( \frac{dy}{dx} \right)^2 = \frac{a^2 y^2}{1-x^2} \]
\[ \Rightarrow \left(1-x^2\right) \left( \frac{dy}{dx} \right)^2 = a^2 y^2 \]
\[ \left(1-x^2\right) \frac{dy}{dx} = a^2 y^2 \]
Again differentiating both sides with respect to \( x \), we obtain
\[ \left( \frac{dy}{dx} \right)^2 \frac{d}{dx} \left(1-x^2\right) + \left(1-x^2\right) \frac{d}{dx} \left( \frac{dy}{dx} \right)^2 = a^2 \frac{d}{dx} \left( y^2 \right) \]
\[ \Rightarrow \left( \frac{dy}{dx} \right)^2 (-2x) + \left(1-x^2\right) \frac{d}{dx} \frac{dy}{dx} \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx} \]
\[ \Rightarrow \left( \frac{dy}{dx} \right)^2 (-2x) + \left(1-x^2\right) \frac{dy}{dx} \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx} \]
\[ \Rightarrow -x \frac{dy}{dx} + \left(1-x^2\right) \frac{d^2 y}{dx^2} = a^2 \cdot y \quad \left[ \frac{dy}{dx} \neq 0 \right] \]
\[ \Rightarrow \left(1-x^2\right) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0 \]
Hence, proved.