CHAPTER 4

QUADRATIC EQUATIONS

(A) Main Concepts and Results

- Quadratic equation : A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.
- Roots of a quadratic equation : A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$.
- The roots of the quadratic equation $ax^2 + bx + c = 0$ are the same as the zeroes of the quadratic polynomial $ax^2 + bx + c$.
- Finding the roots of a quadratic equation by the method of factorisation : If we can factorise the quadratic polynomial $ax^2 + bx + c$, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating to zero the linear factors of $ax^2 + bx + c$.
- Finding the roots of a quadratic equation by the method of completing the square : By adding and subtracting a suitable constant, we club the x^2 and x terms in the quadratic equation so that they become a complete square, and solve for x.
- Quadratic Formula : If $b^2 - 4ac \geq 0$, then the real roots of the quadratic equation $ax^2 + bx + c = 0$ are given by $\frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.
- The expression $b^2 - 4ac$ is called the discriminant of the quadratic equation.
- Existence of roots of a quadratic equation: A quadratic equation $ax^2+bx+c=0$ has

(i) two distinct real roots if $b^2 - 4ac > 0$
(ii) two equal real roots if $b^2 - 4ac = 0$
(iii) no real roots if $b^2 - 4ac < 0$.

(B) Multiple Choice Questions

Choose the correct answer from the given four options:

Sample Question 1 : Which one of the following is not a quadratic equation?

(A) $(x + 2)^2 = 2(x + 3)$ (B) $x^2 + 3x = (-1)(1 - 3x)^2$

(C) $(x + 2)(x - 1) = x^2 - 2x - 3$ (D) $x^3 - x^2 + 2x + 1 = (x + 1)^3$

Solution : Answer (C)

Sample Question 2 : Which constant should be added and subtracted to solve the quadratic equation $4x^2 - \sqrt{3}x - 5 = 0$ by the method of completing the square?

(A) $\frac{9}{16}$ (B) $\frac{3}{16}$ (C) $\frac{3}{4}$ (D) $\frac{\sqrt{3}}{4}$

Solution : Answer (B)

EXERCISE 4.1

Choose the correct answer from the given four options in the following questions:

1. Which of the following is a quadratic equation?

(A) $x^2 + 2x + 1 = (4 - x)^2 + 3$ (B) $-2x^2 = (5 - x)\left(2x - \frac{2}{5}\right)$

(C) $(k + 1)x^2 + \frac{3}{2}x = 7$, where $k = -1$ (D) $x^3 - x^2 = (x - 1)^3$

2. Which of the following is not a quadratic equation?

(A) $2(x - 1)^2 = 4x^2 - 2x + 1$ (B) $2x - x^2 = x^2 + 5$

(C) $(\sqrt{2}x + \sqrt{3})^2 + x^2 = 3x^2 - 5x$ (D) $(x^2 + 2x)^2 = x^4 + 3 + 4x^3$

3. Which of the following equations has 2 as a root?

(A) $x^2 - 4x + 5 = 0$ (B) $x^2 + 3x - 12 = 0$

(C) $2x^2 - 7x + 6 = 0$ (D) $3x^2 - 6x - 2 = 0$

4. If $\frac{1}{2}$ is a root of the equation $x^2 + kx - \frac{5}{4} = 0$, then the value of k is

(A) 2 (B) -2 (C) $\frac{1}{4}$ (D) $\frac{1}{2}$

5. Which of the following equations has the sum of its roots as 3?

(A) $2x^2 - 3x + 6 = 0$ (B) $-x^2 + 3x - 3 = 0$

(C) $\sqrt{2}x^2 - \frac{3}{\sqrt{2}}x + 1 = 0$ (D) $3x^2 - 3x + 3 = 0$

6. Values of k for which the quadratic equation $2x^2 - kx + k = 0$ has equal roots is

(A) 0 only (B) 4 (C) 8 only (D) 0, 8

7. Which constant must be added and subtracted to solve the quadratic equation $9x^2 + \frac{3}{4}x - \sqrt{2} = 0$ by the method of completing the square?

(A) $\frac{1}{8}$ (B) $\frac{1}{64}$ (C) $\frac{1}{4}$ (D) $\frac{9}{64}$

8. The quadratic equation $2x^2 - \sqrt{5}x + 1 = 0$ has

(A) two distinct real roots (B) two equal real roots

(C) no real roots (D) more than 2 real roots

9. Which of the following equations has two distinct real roots?

(A) $2x^2 - 3\sqrt{2}x + \frac{9}{4} = 0$ (B) $x^2 + x - 5 = 0$

(C) $x^2 + 3x + 2\sqrt{2} = 0$ (D) $5x^2 - 3x + 1 = 0$

10. Which of the following equations has no real roots?

(A) $x^2 - 4x + 3\sqrt{2} = 0$ (B) $x^2 + 4x - 3\sqrt{2} = 0$

(C) $x^2 - 4x - 3\sqrt{2} = 0$ (D) $3x^2 + 4\sqrt{3}x + 4 = 0$

11. $(x^2 + 1)^2 - x^2 = 0$ has

(A) four real roots (B) two real roots

(C) no real roots (D) one real root.

(C) Short Answer Questions with Reasoning

Sample Question 1 : Does $(x - 1)^2 + 2(x + 1) = 0$ have a real root? Justify your answer.

Solution : No, since the equation is simplified to $x^2 + 3 = 0$ whose discriminant is -12.

Sample Question 2 : Is the following statement 'True' or 'False'?Justify your answer. If in a quadratic equation the coefficient of x is zero, then the quadratic equation has no real roots.

Solution : False, since the discriminant in this case is $-4ac$ which can still be non-negative if a and c are of opposite signs or if one of a or c is zero.

EXERCISE 4.2

1. State whether the following quadratic equations have two distinct real roots. Justify your answer.

(i) $x^2 - 3x + 4 = 0$ (ii) $2x^2 + x - 1 = 0$

(iii) $2x^2 - 6x + \frac{9}{2} = 0$ (iv) $3x^2 - 4x + 1 = 0$

(v) $(x + 4)^2 - 8x = 0$ (vi) $(x - \sqrt{2})^2 - 2(x + 1) = 0$

(vii) $\sqrt{2}x^2 - \frac{3}{\sqrt{2}}x + \frac{1}{\sqrt{2}} = 0$ (viii) $x(1 - x) - 2 = 0$

(ix) $(x - 1)(x + 2) + 2 = 0$ (x) $(x + 1)(x - 2) + x = 0$

2. Write whether the following statements are true or false. Justify your answers.

(i) Every quadratic equation has exactly one root.

(ii) Every quadratic equation has at least one real root.

(iii) Every quadratic equation has at least two roots.

(iv) Every quadratic equations has at most two roots.

(v) If the coefficient of x^2 and the constant term of a quadratic equation have opposite signs, then the quadratic equation has real roots.

(vi) If the coefficient of x^2 and the constant term have the same sign and if the coefficient of x term is zero, then the quadratic equation has no real roots.

3. A quadratic equation with integral coefficient has integral roots. Justify your answer.

4. Does there exist a quadratic equation whose coefficients are rational but both of its roots are irrational? Justify your answer.

5. Does there exist a quadratic equation whose coefficients are all distinct irrationals but both the roots are rationals? Why?

6. Is 0.2 a root of the equation $x^2 - 0.4 = 0$? Justify.

7. If $b = 0$, $c < 0$, is it true that the roots of $x^2 + bx + c = 0$ are numerically equal and opposite in sign? Justify.

(D) Short Answer Questions

Sample Question 1 : Find the roots of the quadratic equation $2x^2 - \sqrt{5}x - 2 = 0$ using the quadratic formula.

Solution : $b^2 - 4ac = 5 - 4 \times 2 \times (-2) = 21$

Therefore, the roots are $\frac{\sqrt{5} \pm \sqrt{21}}{4}$, i.e., $\frac{\sqrt{5} + \sqrt{21}}{4}$ and $\frac{\sqrt{5} - \sqrt{21}}{4}$

Sample Question 2 : Find the roots of $6x^2 - \sqrt{2}x - 2 = 0$ by the factorisation of the corresponding quadratic polynomial.

Solution :
$$6x^2 - \sqrt{2}x - 2 = 6x^2 - 3\sqrt{2}x + 2\sqrt{2}x - 2$$
$$= 3x(2x - \sqrt{2}) + \sqrt{2}(2x - \sqrt{2})$$
$$= (3x + \sqrt{2})(2x - \sqrt{2})$$

Now, $6x^2 - \sqrt{2}x - 2 = 0$ gives $(3x + \sqrt{2})(2x - \sqrt{2}) = 0$, i.e., $3x + \sqrt{2} = 0$ or $2x - \sqrt{2} = 0$

So, the roots are $-\frac{\sqrt{2}}{3}$ and $\frac{\sqrt{2}}{2}$.

EXERCISE 4.3

1. Find the roots of the quadratic equations by using the quadratic formula in each of the following:

(i) $2x^2 - 3x - 5 = 0$ (ii) $5x^2 + 13x + 8 = 0$

(iii) $-3x^2 + 5x + 12 = 0$ (iv) $-x^2 + 7x - 10 = 0$

(v) $x^2 + 2\sqrt{2}\,x - 6 = 0$ (vi) $x^2 - 3\sqrt{5}\,x + 10 = 0$

(vii) $\frac{1}{2}x^2 - \sqrt{11}\,x + 1 = 0$

2. Find the roots of the following quadratic equations by the factorisation method:

(i) $2x^2 + \frac{5}{3}x - 2 = 0$ (ii) $\frac{2}{5}x^2 - x - \frac{3}{5} = 0$

(iii) $3\sqrt{2}x^2 - 5x - \sqrt{2} = 0$ (iv) $3x^2 + 5\sqrt{5}x - 10 = 0$

(v) $21x^2 - 2x + \frac{1}{21} = 0$

(E) Long Answer Questions

Sample Question 1 : Check whether the equation $6x^2 - 7x + 2 = 0$ has real roots, and if it has, find them by the method of completing the squares.

Solution : The discriminant $= b^2 - 4ac = 49 - 4 \times 6 \times 2 = 1 > 0$

So, the given equation has two distinct real roots.

Now, $6x^2 - 7x + 2 = 0$

i.e., $36x^2 - 42x + 12 = 0$

i.e., $6x - \frac{7}{2}^2 + 12 - \frac{49}{4} = 0$

i.e., $6x - \frac{7}{2}^2 - \frac{1}{2}^2 = 0$ or $\left(6x - \frac{7}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

The roots are given by $6x - \frac{7}{2} = \pm\frac{1}{2}$

i.e., $6x = 4, 3$

i.e., $x = \frac{2}{3}, \frac{1}{2}$.

Sample Question 2 : Had Ajita scored 10 more marks in her mathematics test out of 30 marks, 9 times these marks would have been the square of her actual marks. How many marks did she get in the test?

Solution : Let her actual marks be x

Therefore, $9(x+10) = x^2$

i.e., $x^2 - 9x - 90 = 0$

i.e., $x^2 - 15x + 6x - 90 = 0$

i.e., $x(x-15) + 6(x-15) = 0$

i.e., $(x+6)(x-15) = 0$

Therefore, $x = -6$ or $x = 15$

Since x is the marks obtained, $x \neq -6$. Therefore, $x = 15$.

So, Ajita got 15 marks in her mathematics test.

Sample Question 3 : A train travels at a certain average speed for a distance of 63 km and then travels a distance of 72 km at an average speed of 6 km/h more than its original speed. If it takes 3 hours to complete the total journey, what is its original average speed?

Solution : Let its original average speed be x km/h. Therefore,

$$\frac{63}{x} + \frac{72}{x+6} = 3$$

i.e., $$\frac{7}{x} + \frac{8}{x+6} = \frac{3}{9} = \frac{1}{3}$$

i.e., $$\frac{7(x+6)+8x}{x(x+6)} = \frac{1}{3}$$

i.e., $21(x + 6) + 24x = x(x + 6)$

i.e., $21x + 126 + 24x = x^2 + 6x$

i.e., $x^2 - 39x - 126 = 0$

i.e., $(x + 3)(x - 42) = 0$

i.e., $x = -3$ or $x = 42$

Since x is the average speed of the train, x cannot be negative.

Therefore, $x = 42$.

So, the original average speed of the train is 42 km/h.

EXERCISE 4.4

1. Find whether the following equations have real roots. If real roots exist, find them.

(i) $8x^2 + 2x - 3 = 0$

(ii) $-2x^2 + 3x + 2 = 0$

(iii) $5x^2 - 2x - 10 = 0$

(iv) $\frac{1}{2x-3} + \frac{1}{x-5} = 1, x \neq \frac{3}{2}, 5$

(v) $x^2 + 5\sqrt{5}x - 70 = 0$

2. Find a natural number whose square diminished by 84 is equal to thrice of 8 more than the given number.

3. A natural number, when increased by 12, equals 160 times its reciprocal. Find the number.

4. A train, travelling at a uniform speed for 360 km, would have taken 48 minutes less to travel the same distance if its speed were 5 km/h more. Find the original speed of the train.

5. If Zeba were younger by 5 years than what she really is, then the square of her age (in years) would have been 11 more than five times her actual age. What is her age now?

6. At present Asha's age (in years) is 2 more than the square of her daughter Nisha's age. When Nisha grows to her mother's present age, Asha's age would be one year less than 10 times the present age of Nisha. Find the present ages of both Asha and Nisha.

7. In the centre of a rectangular lawn of dimensions 50 m × 40 m, a rectangular pond has to be constructed so that the area of the grass surrounding the pond would be 1184 m^2 [see Fig. 4.1]. Find the length and breadth of the pond.

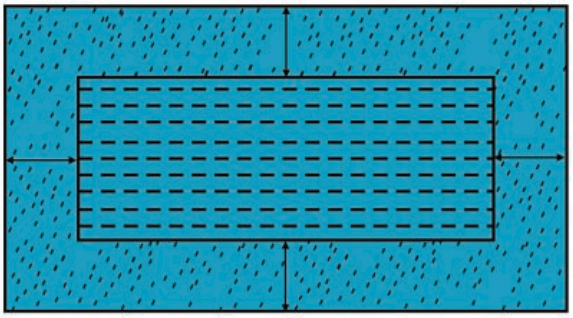


Fig. 4.1

8. At t minutes past 2 pm, the time needed by the minutes hand of a clock to show 3 pm was found to be 3 minutes less than $\frac{t^2}{4}$ minutes. Find t.