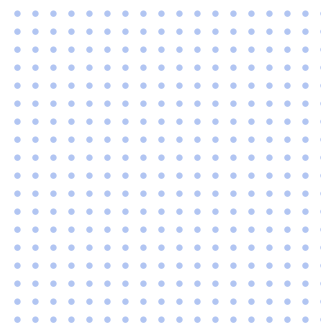




FOR ISI ADMISSION TEST EXAM PREPARATION

ISI Admission Test 2020 Question Paper · M.Math PMA

EXAM

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YEAR

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TYPE

Question Paper

SUBJECT

M.Math PMA

Notation. In the following, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \dots\}$. For an element g of a group G , $o(g)$ denotes the order of g in G . For a set A , $|A|$ denotes its cardinality.

(1) Let V be a finite dimensional vector space and let W be a proper subspace of V . Let W' be another subspace of V such that $V = W \oplus W'$, i.e. $V = \text{Span}(W \cup W')$ and $W \cap W' = \{0\}$. Let $T : V \rightarrow V$ be an invertible linear map such that $T(W) \subset W$. Which of the following statements is necessarily true?

- (a) $T(W') \subset W'$.
- (b) $W' \subset T(W')$.
- (c) $T(W') \cap W = \{0\}$.
- (d) $W' \subset \text{Ker}(T)$.

(2) Let V be a finite dimensional real vector space, and let $T : V \rightarrow V$ be a linear map such that $\text{Range}(T) = \text{Ker}(T)$. Which of the following statements is not necessarily true?

- (a) $T = 0$.
- (b) $T^2 = 0$.
- (c) 0 is an eigenvalue of T .
- (d) All eigenvalues of T are equal to 0.

(3) Consider the vector space $\mathbb{R}^n$ equipped with the Euclidean metric d defined by

$$d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

Let W be a proper subspace of $\mathbb{R}^n$. Which of the following statements is necessarily true?

- (a) W is closed.
- (b) W is open.
- (c) W is not closed.
- (d) W is neither closed nor open.

(4) Let A be a 5×5 real matrix. If $A = (a_{ij})$, let A_{ij} denote the cofactor of the entry a_{ij} , for $1 \leq i, j \leq 5$. Let $\hat{A}$ denote the matrix whose (i, j) -th entry is A_{ij} , $1 \leq i, j \leq 5$. Suppose the rank of A is 3. What is the rank of $\hat{A}$?

- (a) 1
- (b) 3
- (c) 5
- (d) 0

(5) For $n \geq 2$, the determinant of the $n \times n$ permutation matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & \dots & \dots \\ 1 & 0 & \dots & \dots & \dots \end{pmatrix}$$

is

- (a) $(-1)^n$
- (b) $(-1)^{n(n-1)/2}$
- (c) -1
- (d) 1

(6) Let $M_2(\mathbb{R})$ denote the vector space of all 2×2 matrices over the field of real numbers, i.e.

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Let $S \subset M_2(\mathbb{R})$ be the subspace defined by

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + c = 0 \right\}.$$

Then the dimension of S is

- (a) 1
- (b) 2
- (c) 3
- (d) 4

(7) Let V be a finite dimensional real vector space of dimension $n > 1$ and let $W \subset V$ be a subspace of dimension $n - 1$. A linear map from V to $\mathbb{R}$ is called a linear functional on V . Which of the following statements is necessarily true?

- (a) There does not exist any linear functional on V such that W is the kernel of that linear functional.
- (b) W is the kernel of a unique linear functional on V .
- (c) W is the kernel of a linear functional on V .
- (d) There exists a non-zero linear functional on V whose kernel strictly contains W .

(8) Let $\langle \cdot, \cdot \rangle_n$ denote the standard inner product on the vector space $\mathbb{R}^n$, i.e.

$$\langle x, y \rangle_n = \sum_{i=1}^n x_i y_i$$

for vectors $x, y \in \mathbb{R}^n$. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map such that

$$\langle Tx, Ty \rangle_m = \langle x, y \rangle_n$$

for all $x, y \in \mathbb{R}^n$. Which of the following statements is necessarily true?

- (a) $n \geq m$.
- (b) $n \leq m$.
- (c) $n = m$.
- (d) The map T is onto.

(9) Let V be a finite dimensional real inner product space, and let $T : V \rightarrow V$ be a linear map such that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$. Suppose $W \subset V$ is a proper subspace of V such that $T(W) \subset W$. Define a subspace $W^\perp$ of V by

$$W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

Which of the following statements is necessarily true?

- (a) $T(W^\perp)$ is not contained in $W^\perp$.
- (b) $T(W^\perp)$ is contained in $W^\perp$.
- (c) $T(W^\perp) \cap W^\perp = \{0\}$.
- (d) $W^\perp$ is contained in $T(W^\perp)$.

(10) Let R be a ring with unit such that $a^2 = a$ for all $a \in R$. Which of the following statements is not necessarily true?

- (a) $ab = -ba$ for all $a, b \in R$.
- (b) $a = -a$ for all $a \in R$.
- (c) R is commutative.
- (d) $R = \{0, 1\}$.

(11) Let S be a nonempty set, and let $P(S)$ be the power set of S , i.e. $P(S) = \{A \mid A \subset S\}$. Define a binary operation Δ on $P(S)$ by $A \Delta B := (A \cup B) \setminus (A \cap B)$ for $A, B \in P(S)$. Which of the following statements is necessarily true?

- (a) $(P(S), \Delta)$ is not a group as Δ is not associative.
- (b) $(P(S), \Delta)$ is not a group as there is no identity.
- (c) $(P(S), \Delta)$ is an abelian group.
- (d) $(P(S), \Delta)$ is a non-abelian group.

(12) Let I_1, I_2 be ideals of a commutative ring R . Define the set

$$I_1 + I_2 := \{a + b \mid a \in I_1, b \in I_2\}.$$

Which of the following statements is not necessarily true?

- (a) $I_1 + I_2$ is an ideal of R .
- (b) $I_1 \subset I_1 + I_2$.
- (c) $|I_1 + I_2| = |I_1| + |I_2|$ if R is finite.
- (d) $|I_1 + I_2| = |I_1| \cdot |I_2|$ if $I_1 \cap I_2 = \{0\}$ and R is finite.

(13) Let I be an ideal of a commutative ring R . Define the set

$$\sqrt{I} := \{a \in R \mid \text{There exists } n \geq 1 \text{ such that } a^n \in I\}.$$

Which of the following statements is necessarily true?

- (a) $\sqrt{I}$ is an ideal.
- (b) $\sqrt{I}$ is not an ideal.
- (c) $\sqrt{I} = I$.
- (d) $\sqrt{I} \subset I$.

(14) How many non-isomorphic groups are there of order 15?

- (a) 1
- (b) 2
- (c) 3
- (d) 5

(15) Suppose G is a group and $a, b \in G$. Which of the following statements is necessarily true?

- (a) There exists a positive integer n such that $a^n = b^n$.
- (b) $(ab)^{-1} = a^{-1}b^{-1}$.
- (c) $o(ab) = o(ba)$.
- (d) None of the above statements is necessarily true.

(16) Let H_1, H_2 be distinct subgroups of a finite abelian group G . Define the subgroup H_1H_2 by $H_1H_2 = \{h_1h_2 \mid h_1 \in H_1, h_2 \in H_2\}$.

Which of the following statements is necessarily true?

- (a) $|G| \leq |H_1| + |H_2|$.
- (b) $|G/(H_1 \cap H_2)| = |G/H_1| \cdot |G/H_2|$.
- (c) $|G/H_1| = |G/H_2| \cdot |H_2/(H_1 \cap H_2)|$.
- (d) $|(H_1H_2)/H_1| = |H_2/(H_1 \cap H_2)|$.

(17) Let $p > 3$ be a prime number and let S_p denote the symmetric group on p symbols. How many p -Sylow subgroups are there in S_p ?

- (a) 1.
- (b) p .
- (c) 2.
- (d) $(p-2)!$.

(18) Let R be a commutative ring with unit and let

$$N = \{a \in R \mid a^n = 0 \text{ for some integer } n \geq 0\}.$$

Which of the following statements is necessarily true?

- (a) Any prime ideal of R contains N .
- (b) N is not an ideal.
- (c) N is a prime ideal.
- (d) $N = \{0\}$.

(19) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable, and suppose $\lim_{x \rightarrow \infty} f''(x) = 1$. Which of the following statements is necessarily true?

- (a) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^2} = 1$.
- (b) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^2}$ does not exist.
- (c) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^2} = 2$.
- (d) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^2} = 1/2$.

(20) Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined by $f_n(x) = (\cos(\pi x))^{2n}$. Which of the following statements is true?

- (a) The sequence $\{f_n\}$ converges uniformly on $[0, 1]$.
- (b) The sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to a function f such that f has exactly one point of discontinuity.
- (c) The sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to a function f such that f has exactly two points of discontinuity.
- (d) The sequence $\{f_n\}$ does not converge pointwise on $[0, 1]$.

(21) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function such that

$$\int_{-1}^1 f(x)x^{2n}dx = 0$$

for all $n \geq 0$. Which of the following statements is necessarily false?

- (a) $\int_{-1}^1 f(x)^2 dx = \int_{-1}^1 f(-x)^2 dx$.
- (b) $\left(\sup_{x \in [-1, 1]} f(x)\right) + \left(\inf_{x \in [-1, 1]} f(x)\right) = 0$.
- (c) $f(0) \neq 0$.
- (d) $f(1/2)f(-1/2) \leq 0$.

(22) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$. Which of the following statements is necessarily false?

- (a) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^{1+\epsilon}} = 0$ for all $\epsilon > 0$.
- (b) $\lim_{x \rightarrow \infty} \frac{f(x)}{x}$ does not exist.
- (c) $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 1$.
- (d) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^{1-\epsilon}} = +\infty$ for all $\epsilon > 0$.

(23) Let $\{0, 1\}^{\mathbb{N}}$ denote the set of all sequences $\{x_n\}$ such that $x_n \in \{0, 1\}$ for all $n \geq 1$. Define a map $f : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ by

$$f(\{x_n\}) := \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Which of the following statements is true?

- (a) The map f is one-to-one and onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1]$.
- (b) The map f is one-to-one and onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1)$.
- (c) The map f is onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1]$ and $|f^{-1}(1/2)| = 2$.
- (d) The map f is onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1]$ and $|f^{-1}(1)| = 2$.

(24) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a monotone increasing function, and define $f_n : [0, \infty) \rightarrow \mathbb{R}$ by

$$f_n(x) = f(x+n), \quad x \in [0, \infty)$$

for all $n \geq 1$. Suppose that for some $x_0 \in [0, \infty)$, the limit $\lim_{n \rightarrow \infty} f_n(x_0)$ exists. Which of the following statements is necessarily false?

- (a) The sequence $\{f_n\}$ converges pointwise on $[0, \infty)$.
- (b) The sequence $\{f_n\}$ converges uniformly on $[0, \infty)$.
- (c) The limit $\lim_{x \rightarrow \infty} f(x)$ exists.
- (d) The function f is unbounded on $[0, \infty)$.

(25) Let X, Y be sets and let $f : X \rightarrow Y$ be a function. Let $\{S_i\}_{i \in I}$ be a family of subsets of X , i.e. $S_i \subset X$ for all $i \in I$, where I is an index set. Which of the following statements is not necessarily true?

- (a) $f(\cup_{i \in I} S_i) \subset \cup_{i \in I} f(S_i)$.
- (b) $f(\cup_{i \in I} S_i) \supset \cup_{i \in I} f(S_i)$.
- (c) $f(\cap_{i \in I} S_i) \subset \cap_{i \in I} f(S_i)$.
- (d) $f(\cap_{i \in I} S_i) \supset \cap_{i \in I} f(S_i)$.

(26) Let S be the set of all those nonnegative real numbers α with the following property: if $\{x_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} x_n < +\infty$, then we also have $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n^\alpha} < +\infty$. Which of the following statements is true?

- (a) $S = \emptyset$.
- (b) $S \supset (1/4, \infty)$.
- (c) $S \supset (1/2, \infty)$.
- (d) $S \subset (3/4, \infty)$.

(27) Let X be a finite set. Let $P(X)$ be the power set of X , i.e. the set whose elements are all subsets of X . Which of the following defines a metric on the power set $P(X)$?

- (a) $d(V, W) = |(V \cup W) \setminus (V \cap W)|$.
- (b) $d(V, W) = |V \cap W|$.
- (c) $d(V, W) = |V \setminus W|$.
- (d) $d(V, W) = |V \cup W|$.

(28) The tangent line to the curve $2x^6 + y^4 = 9xy$ at the point $(1, 2)$ has slope

- (a) $3/23$.
- (b) $6/23$.
- (c) $9/23$.
- (d) $4/7$.

(29) Consider the following statements:

- (i) If $\sum_n a_n$ and $\sum_n b_n$ are convergent, then $\sum_n a_n b_n$ is convergent.
- (ii) If $\sum_n a_n$ is convergent and $\sum_n b_n$ is absolutely convergent, then $\sum_n a_n b_n$ is absolutely convergent.
- (iii) If $a_n \geq 0$ for all n , $\sum_n a_n$ is convergent, and $\{b_n\}$ is a bounded sequence, then $\sum_n a_n b_n$ is absolutely convergent.

Which of the following statements is true?

- (a) All of the statements (i), (ii), (iii) are true.
- (b) Statement (ii) is true but statement (iii) is false.
- (c) Only statements (i) and (iii) are true.
- (d) Only statements (ii) and (iii) are true.

(30) Consider the metric space $(\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}, d)$, where the metric d is defined by $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for $m, n \in \mathbb{N}$, and $d(n, \infty) = 1/n$ for $n \in \mathbb{N}$. Let $f : \overline{\mathbb{N}} \rightarrow \mathbb{R}$ be a continuous function between metric spaces (where $\mathbb{R}$ is equipped with its usual metric). Which of the following statements is necessarily false?

- (a) The metric space $\overline{\mathbb{N}}$ is compact.
- (b) The function f is unbounded.
- (c) The function f is uniformly continuous.
- (d) For any $x \in \mathbb{R}$, the set $f^{-1}(\{x\})$ is compact.