

NCERT SOLUTIONS

CLASS - 12th



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Class : 12th
Subject : Maths
Chapter : 5
Chapter Name : Continuity and Differentiability

Q1 Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$

Answer.

The given function is $f(x) = 5x - 3$

$$\text{At } x = 0, f(0) = 5 \times 0 - 3 = -3$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (5x - 3) = 5 \times 0 - 3 = -3$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

$$\text{At } x = -3, f(-3) = 5 \times (-3) - 3 = -18$$

$$\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} (5x - 3) = 5 \times (-3) - 3 = -18$$

$$\therefore \lim_{x \rightarrow -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

$$\text{At } x = 5, f(5) = 5 \times 5 - 3 = 25 - 3 = 22$$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (5x - 3) = 5 \times 5 - 3 = 22$$

$$\therefore \lim_{x \rightarrow 5} f(x) = f(5)$$

Therefore, f is continuous at $x = 5$

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Q2 Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$

Answer.

The given function is $f(x) = 2x^2 - 1$

$$\text{At } x = 3, f(3) = 2 \times 3^2 - 1 = 17$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (2x^2 - 1) = 2 \times 3^2 - 1 = 17$$

$$\therefore \lim_{x \rightarrow 3} f(x) = f(3)$$

Thus, f is continuous at $x = 3$

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Processing math: 100% Cving functions for continuity.

$$(a)f(x) = x - 5$$

$$(b) f(x) = \frac{1}{x-5}, x \neq 5$$

$$(c)f(x) = \frac{x^2-25}{x+5}, x \neq -5$$

$$(d)f(x) = |x - 5|$$

Answer.

(a) The given function is $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is $k - 5$.

It is also observed that, $f(x) = \lim_{x \rightarrow k}(x - 5) = k - 5 = f(k)$

$$\therefore \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, f is continuous at every real number and therefore, it is a continuous function.

$$(b)f(x) = \frac{1}{x-5}, x \neq 5$$

For any real number $k \neq 5$, we obtain

$$\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} \frac{1}{x-5} = \frac{1}{k-5}$$

$$\text{Also, } f(k) = \frac{1}{k-5} \quad (\text{As } k \neq 5)$$

$$\therefore \lim_{x \rightarrow k} f(x) = f(k)$$

$$\therefore \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

$$(c) \text{ The given function is } f(x) = \frac{x^2-25}{x+5}, x \neq -5$$

For any real number $c \neq -5$, we obtain

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{x^2-25}{x+5} = \lim_{x \rightarrow c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \rightarrow c} (x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

$$(d) \text{ The given function } f(x) = |x - 5| = \begin{cases} 5 - x, & \text{if } x < 5 \\ x - 5, & \text{if } x \geq 5 \end{cases}$$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case 1: $c < 5$

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$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (5 - x) = 5 - c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5 .

Case II : $c = 5$

$$\text{Then, } f(c) = f(5) = (5 - 5) = 0$$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (5 - x) = (5 - 5) = 0$$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (x - 5) = 0$$

$$\therefore \lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at $x = 5$

Case III: $c > 5$

$$\text{Then, } f(c) = f(5) = c - 5$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow \infty} (x - 5) = c - 5$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at every real numbers greater than 5 .

Hence, f is continuous at every real number and therefore, it is a continuous function.

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Q4 Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Answer.

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n , and its value at n is n^n .

$$\text{Then, } \lim_{x \rightarrow m} f(n) = \lim_{x \rightarrow m} (x^n) = n^n$$

$$\therefore \lim_{x \rightarrow n} f(x) = f(n)$$

Therefore, f is continuous at n , where n is a positive integer.

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Q5

Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer. The given function f is

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

Processing math: 100%

It is evident that f is defined at 0 and its value at 0 is 0 .

$$\text{Then, } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

At $x = 1$

$$f \text{ is defined at 1 and its value at 1 is 1 . } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (5) = 5$$

$$\text{The left hand limit of } f \text{ at } x = 1 \text{ is, } \therefore \lim_{x \rightarrow 1} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x = 1$$

The right hand limit of f at $x = 1$ is,

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5 .

$$\text{Then, } \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (5) = 5$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$

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Q6 Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}$$

$$\text{Answer. The given function } f \text{ is } f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}$$

It is evident that the given function f is defined at all the points of the real line. Let c be a point on the real line. Then, three cases arise

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

$$\text{Then, } f(c) = 2c + 3$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x + 3) = 2c + 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 2$

case (ii) $c > 2$

$$\text{Then, } f(c) = 2c - 3$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x - 3) = 2c - 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

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Therefore, f is continuous at all points x , such that $x > 2$

case (iii) $c = 2$

Then, the left hand limit of f at $x = 2$ is,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 3) = 2 \times 2 + 3 = 7$$

The right hand limit of f at $x = 2$ is,

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (2x - 3) = 2 \times 2 - 3 = 1$$

It is observed that the left and right hand limit of f at $x = 2$ do not coincide. Therefore, f is not continuous at $x = 2$. Hence, $x = 2$ is the only point of discontinuity of f .

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Q7 Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} |x| + 3 = -x + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

The given function f is defined at all the points of the real line. Let c be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow \infty} (-x + 3) = -c + 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Case II:

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \rightarrow -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2x) = -2c$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (-2x) = -2 \times 3 = -6$$

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The right hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^+} f(x) = \lim (6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case v:

$$\text{If } c > 3, \text{ then } f(c) = 6c + 2 \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (6x + 2) = 6c + 2$$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f .

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Q8 Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line. Let c be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = -1$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow \infty} (-1) = -1$$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-1) = -1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1) = 1$$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Case III:

If $c > 0$, then $f(c) = 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} e(1) = 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f .

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Q9 Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$$f(c) = -1 = \lim_{x \rightarrow c} f(x)$$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

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Q10 Find all points of discontinuity of f , where f is defined by

Processing math: 100%

$$f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}$$

The given function f is defined at all the points of the real line. Let c be a point on the real line.

Case I:

$$\text{If } c < 1, \text{ then } f(c) = c^2 + 1 \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2 + 1) = c^2 + 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$

Case II:

$$\text{If } c = 1, \text{ then } f(c) = f(1) = 1 + 1 = 2$$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 1^2 + 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 1 + 1 = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, f is continuous at $x = 1$

Case III:

$$\text{If } c > 1, \text{ then } f(c) = c + 1$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 1) = c + 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$. Hence, the given function f has no point of discontinuity.

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Q11 Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

Processing math: 100% f is defined at all the points of the real line. Let c be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow c} (x^3 - 3) = c^3 - 3$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x^3 - 3) = 2^3 - 3 = 5$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x^2 + 1) = 2^2 + 1 = 5$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2^+} (x^3 - 3) = 2^3 - 3 = 5$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + 1) = 2^2 + 1 = 5$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$

If $c > 2$, then $f(c) = c^2 + 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2 + 1) = c^2 + 1$$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 2$. Thus, the given function f is continuous at every point on the real line. Hence, f has no point of discontinuity.

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Q12 Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

The given function f is defined at all the points of the real line. Let c be a point on the real line.

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Case I:

If $c < 1$, then $f(c) = c^{10} - 1$ and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (x^{10} - 1) = c^{10} - 1$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2) = 1^2 = 1$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide. Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c^2$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2) = c^2$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$ Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f .

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Q13 Is the function defined by

$$f(x) = \begin{cases} x + 5, & \text{if } x \leq 1 \\ x - 5, & \text{if } x > 1 \end{cases} \text{ a continuous function ?}$$

Answer. The given function is

$$f(x) = \begin{cases} x + 5, & \text{if } x \leq 1 \\ x - 5, & \text{if } x > 1 \end{cases}$$

The given function f is defined at all the points of the real line. Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 5$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 5) = c + 5$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$

Case II:

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 5) = 1 + 5 = 6$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 5) = 1 - 5 = -4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide. Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c - 5$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - 5) = c - 5$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$. Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f .

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Q14 Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

Answer. The given function is

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

The given function is defined at all points of the interval $[0, 10]$. Let c be a point in the interval $[0, 10]$.

Case I

If $0 \leq c < 1$, then $f(c) = 3$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (3) = 3$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in the interval $[0, 1)$.

Case II:

If $c = 1$, then $f(1) = 3$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3) = 3$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4) = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide. Therefore, f is not

Case III:

If $1 < c < 3$, then $f(c) = 4$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (4) = 4$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(1, 3)$.

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (5) = 5$$

It is observed that the left and right hand limits of f at $x = 3$ do not coincide.

Therefore f is not continuous at $x = 3$

Case V:

If $3 < c \leq 10$, then $f(c) = 5$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (5) = 5$

$$\lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(3, 10)$

Hence, f is not continuous at $x = 1$ and $x = 3$

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Q15 Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer. The give function is

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

The given function is defined at all points of the real line. Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x) = 2c$$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 0$

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x) = 2 \times 0 = 0$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (0) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \rightarrow \infty} f(x) = \lim (0) = 0$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(0, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim (0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide. Therefore, f is not continuous at $x = 1$

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (4x) = 4c$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$

Hence, f is not continuous only at $x = 1$

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Q16 Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Answer. The given function is

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Processing math: 100% defined at all points of the real line. Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2) = -2$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < -1$

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (-2) = -2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2x) = 2 \times (-1) = -2$$

$$\therefore \lim_{x \rightarrow -1} f(x) = f(-1)$$

Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (2x) = 2c$ Therefore, f is continuous at all points of the interval $(-1, 1)$

$$\therefore \lim f(x) = f(c)$$

Case IV

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (2x) = 2 \times 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} 2 = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(c)$$

Therefore, f is continuous at $x = 2$

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2) = 2$

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

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Q17 Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases} \text{ is continuous at } x = 3$$

Answer. The given function f is

Processing math: 100%

$$f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

If f is continuous at $x = 3$, then

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

Also,

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (ax + 1) = 3a + 1$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3^+} (bx + 3) = 3b + 3$$

$$f(3) = 3a + 1$$

Therefore, from (1), we obtain

$$3a + 1 = 3b + 3 = 3a + 1$$

$$\Rightarrow 3a + 1 = 3b + 3$$

$$\Rightarrow 3a = 3b + 2$$

$$\Rightarrow a = b + \frac{2}{3}$$

Therefore, the required relationship is given by, $a = b + \frac{2}{3}$

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Q18 For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

Answer. The given function is

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \lambda(x^2 - 2x) = \lim_{x \rightarrow 0^-} (4x + 1) = \lambda(0^2 - 2 \times 0)$$

$$\Rightarrow \lambda(0^2 - 2 \times 0) = 4 \times 0 + 1 = 0$$

$$\Rightarrow 0 = 1 = 0, \text{ which is not possible}$$

Therefore, there is no value of λ for which f is continuous at $x = 0$

At $x = 1$

$$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$$

$$\lim_{x \rightarrow 1} (4x + 1) = 4 \times 1 + 1 = 5$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, for any values of λ , f is continuous at $x = 1$

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Q19

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x .

Answer.

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n - [n] = n - n = 0$$

The left hand limit of f at $x = n$ is,

$$\lim_{x \rightarrow m} g(x) = \lim_{x \rightarrow m} (x - [x]) = \lim_{x \rightarrow 1} (x) - \lim_{x \rightarrow \infty} [x] = n - (n - 1) = 1$$

It is observed that the left and right hand limits of f at $x = n$ do not coincide. Therefore, f is not continuous at $x = n$ Hence, g is discontinuous at all integral points

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Q20 Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = p$?

Answer. The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = p$

$$\text{At } x = \pi, f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$$

$$\text{Consider } \lim_{x \rightarrow x} f(x) = \lim_{x \rightarrow x} (x^2 - \sin x + 5)$$

Put $x = \pi + h$

If $x \rightarrow \pi$, then it is evident that $h \rightarrow 0$

$$\text{Consider } \lim f(x) = \lim_{x \rightarrow \pi} (x^2 - \sin x + 5)$$

Put $x = \pi + h$

If $x \rightarrow \pi$, then it is evident that $h \rightarrow 0$

$$\begin{aligned}
\therefore \lim_{x \rightarrow \pi} f(x) &= \lim_{x \rightarrow \pi} (x^2 - \sin x + 5) \\
&= \lim_{h \rightarrow 0} [(\pi + h)^2 - \sin(\pi + h) + 5] \\
&= \lim_{h \rightarrow 0} (\pi + h)^2 - \lim_{h \rightarrow 0} \sin(\pi + h) + \lim_{h \rightarrow 0} 5 \\
&= (\pi + 0)^2 - \lim_{h \rightarrow 0} [\sin \pi \cosh + \cos \pi \sinh] + 5 \\
&= \pi^2 - \lim_{h \rightarrow 0} \sin \pi \cosh - \lim_{h \rightarrow 0} \cos \pi \sinh + 5 \\
&= \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5 \\
&= \pi^2 - 0 \times 1 - (-1) \times 0 + 5 \\
&= \pi^2 + 5 \\
\therefore \lim_{x \rightarrow \infty} f(x) &= f(\pi)
\end{aligned}$$

Therefore, the given function f is continuous at $x = \pi$

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Q21 Discuss the continuity of the following functions.

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \times \cos x$

Answer.

It is known that if g and h are two continuous functions, then

$g + h$, $g - h$, and $g h$ are also continuous.

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$g(c) = \sin c$$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} \sin x$$

$$= \lim_{h \rightarrow 0} \sin(c + h)$$

$$= \lim_{h \rightarrow 0} [\sin c \cosh + \cos c \sinh]$$

$$= \lim_{h \rightarrow 0} (\sin c \cosh) + \lim_{h \rightarrow 0} (\cos c \sinh)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

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Q22 Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer. It is known that if g and h are two continuous functions, then

(i) $\frac{h(x)}{g(x)}, g(x) \neq 0$ is continuous

(ii) $\frac{1}{g(x)}, g(x) \neq 0$ is continuous

(iii) $\frac{1}{h(x)}, h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$g(c) = \sin c$$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} \sin x$$

$$= \lim_{h \rightarrow 0} \sin(c + h)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} \cos x$$

Processing math: 100%

$$= \lim_{h \rightarrow 0} \cos(c + h)$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} [\cos c \cosh - \sin c \sinh] \\
&= \lim_{h \rightarrow 0} \cos c \cosh - \lim_{h \rightarrow 0} \sin c \sinh \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c \\
\therefore \lim_{x \rightarrow \infty} h(x) &= h(c)
\end{aligned}$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

It can be concluded that,

$$\csc x = \frac{1}{\sin x}, \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \csc x, x \neq n\pi (n \in \mathbb{Z}) \text{ is continuous}$$

$$\csc x = \frac{1}{\sin x}, \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \csc x, x \neq n\pi (n \in \mathbb{Z}) \text{ is continuous}$$

Therefore, cosecant is continuous except at $x = n\pi, n \in \mathbb{Z}$

$$\cot x = \frac{\cos x}{\sin x}, \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \cot x, x \neq n\pi (n \in \mathbb{Z}) \text{ is continuous}$$

Therefore, cotangent is continuous except at $x = n\pi, n \in \mathbb{Z}$

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Q23 Find the points of discontinuity of f , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}$$

It is evident that f is defined at all points of the real line. Let c be a real number.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = \frac{\sin c}{c} \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(\frac{\sin x}{x} \right) = \frac{\sin c}{c}$$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 0$

Case II:

If $c > 0$, then $f(c) = c + 1$ and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (x + 1) = c + 1$

$$\therefore \lim_{x \rightarrow \infty} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x \geq 0$

Case III:

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 1) = 1$$

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

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Q24 Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ is a continuous function?}$$

Answer. The given function f is

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It is evident that f is defined at all points of the real line. Let c be a real number.

If $c \neq 0$, then $f(c) = c^2 \sin \frac{1}{c}$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(x^2 \sin \frac{1}{x} \right) = \left(\lim_{x \rightarrow c} x^2 \right) \left(\lim_{x \rightarrow c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right)$$

Processing math: 100%

 $\sin \frac{1}{x} \leq 1, x \neq 0$

$$\Rightarrow -x^2 \leq \sin \frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \rightarrow 0} (-x^2) \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq \lim_{x \rightarrow 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = 0$$

$$\text{Similarly, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0) = \lim_{x \rightarrow 0} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

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Q25 Examine the continuity of f , where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer. The given function f is

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Case II:

If $c = 0$, then $f(0) = -1$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Processing math: 100%

Q26 Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer. The given function

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$

f is defined at and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

$$\text{Put } x = \frac{\pi}{2} + h$$

$$\text{Then } x \rightarrow \frac{\pi}{2} \Rightarrow h \rightarrow 0$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{2}} f(x) &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} \\ &= k \lim_{h \rightarrow 0} \frac{-\sinh}{-2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sinh}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2} \end{aligned}$$

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow \frac{k}{2} = 6$$

$$\Rightarrow k = 6$$

Processing math: 100% value of k is 6

Q28 Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

Answer. The given function is

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

The given function f is continuous at $x = p$, if f is defined at $x = p$ and if the value of f at $x = p$ equals the limit of f at $x = p$

It is evident that f is defined at $x = p$ and $f(\pi) = k\pi + 1$

$$\lim_{x \rightarrow \pi^+} f(x) = \lim_{x \rightarrow \pi^+} \cos x = f(\pi)$$

$$\Rightarrow \lim_{x \rightarrow \pi^-} (kx + 1) = \lim_{x \rightarrow \pi^-} \cos x = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos \pi = -1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Q29 Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5 \\ 3x - 5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

Answer. The given function f is

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5 \\ 3x - 5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of f at $x = 5$ equals the limit of f at $x = 5$

It is evident that f is defined at $x = 5$ and $f(5) = kx + 1 = 5k + 1$

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} (3x - 5) = f(5)$$

$$\Rightarrow \lim_{x \rightarrow 5^-} (kx + 1) = \lim_{x \rightarrow 5^-} (3x - 5) = 5k + 1$$

$$\Rightarrow 5k + 1 = 15 - 5 = 10$$

$$\Rightarrow 5k + 1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

Q30 Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

is a continuous function.

Answer. The given function f is

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

It is evident that the given function f is defined at all points of the real line. If f is a continuous function, then f is continuous at all real numbers. In particular, f is continuous at $x = 2$ and $x = 10$

Since f is continuous at $x = 2$, we obtain

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} (5) = \lim_{x \rightarrow 2^+} (ax + b) = 5$$

$$\Rightarrow 5 = 2a + b = 5$$

$$\Rightarrow 2a + b = 5$$

$$\lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x \rightarrow 10^-} (ax + b) = \lim_{x \rightarrow 10^+} (21) = 21$$

$$\Rightarrow 10a + b = 21 = 21$$

$$\Rightarrow 10a + b = 21$$

On subtracting equation (1) from equation (2), we obtain $8a = 16 \Rightarrow a = 2$ By putting $a = 2$ in equation (1), we obtain $2 \times 2 + b = 5 \Rightarrow 4 + b = 5$

$$\Rightarrow b = 1$$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

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Q31 Show that the function defined by

$$f(x) = \cos(x^2) \text{ is a continuous function.}$$

Answer.

$$\text{The given function is } f(x) = \cos(x^2)$$

This function f is defined for every real number and f can be written as the composition of two functions as,

$$f = g \circ h, \text{ where } g(x) = \cos x \text{ and } h(x) = x^2$$

$$\left[\begin{array}{l} \text{Processing math: 100\%} \\ \cdot (g \circ h)(x) = g(h(x)) \end{array} \right] = g(x^2) = \cos(x^2) = f(x)$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$\begin{aligned}\lim_{x \rightarrow c} g(x) &= \lim_{x \rightarrow c} \cos x \\ &= \lim_{h \rightarrow 0} \cos(c + h) \\ &= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h \rightarrow 0} \cos c \cos h - \lim_{h \rightarrow 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c \\ \therefore \lim_{x \rightarrow c} g(x) &= g(c)\end{aligned}$$

Therefore, $g(x) = \cos x$ is continuous function.

$$h(x) = x^2$$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$$\lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} x^2 = k^2$$

$$\therefore \lim_{x \rightarrow k} h(x) = h(k)$$

Therefore, h is a continuous function. It is known that for real valued functions g and h , such that $(g \circ h)$ is defined at c , if g is continuous at c and if h is continuous at $g(c)$, then $(g \circ h)$ is continuous at c .

$f(x) = (g \circ h)(x) = \cos(x^2)$ is a continuous function.

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Q32 Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer. The given function is $f(x) = |\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = |x|$ and $h(x) = \cos x$

$$[\because (g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Processing math: 100%

Clearly, g is defined for all real numbers. Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$

case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} x = c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} 0(x) = g(0)$$

Therefore, g is continuous at $x = 0$ From the above three observations, it can be concluded that g is continuous at all points. $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} \cos x$$

$$x \rightarrow c$$

$$= \lim_{h \rightarrow 0} \cos(c + h)$$

$$= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h]$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function. It is known that for real valued functions g and h , such that $(g \circ h)$ is defined at c , if g is continuous at c and if h is continuous at $g(c)$, then $(g \circ h)$ is continuous at c .

$$f(x) = (g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x| \text{ is a continuous function.}$$

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Q33 Examine that $\sin |x|$ is a continuous function. *

Answer. $f(x) = \sin |x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = |x|$ and $h(x) = \sin x$

$$g(\sin x) = |\sin x| = f(x)$$

It has to be proved that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} x = c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} x = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} g(x) = g(0)$$

Therefore, g is continuous at $x = 0$. From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number. Let c be a real number. Put $x = c + k$

If $x \rightarrow c$, then $k \rightarrow 0$ $h(c) = \sin c$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} \sin x$$

$$= \lim_{x \rightarrow c} \sin(c + k)$$

$$= \lim_{k \rightarrow 0} [\sin c \cos k + \cos c \sin k]$$

$$= \lim_{k \rightarrow 0} (\sin c \cos k) + \lim_{k \rightarrow 0} (\cos c \sin k)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

$$\therefore \lim_{x \rightarrow c} h(x) = g(c)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h , such that $(g \circ h)$ is defined at c , if g is continuous at c and if h is continuous at $g(c)$, then $(g \circ h)$ is continuous at c .

$f(x) = (g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Answer. The given function is $f(x) = |x| - |x + 1|$

The two functions, g and h , are defined as

$$g(x) = |x| \text{ and } h(x) = |x + 1|$$

Then, $f = g - h$

The continuity of g and h is examined first.

$g(x) = |x|$ Can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers. Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} x = c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

Therefore, g is continuous at $x = 0$ From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x + 1|$ can be written as

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c + 1)$ and $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow \infty} [-(x + 1)] = -(c + 1)$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, h is continuous at all points x , such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c + 1$ and $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} (x + 1) = c + 1$

$$\therefore \lim_{x \rightarrow \infty} h(x) = h(c)$$

Therefore, h is continuous at all points x , such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1 + 1 = 0$

$$\lim_{x \rightarrow -1} h(x) = \lim_{x \rightarrow -1} [-(x+1)] = -(-1+1) = 0$$

$$\lim_{x \rightarrow -1} -h(x) = \lim_{x \rightarrow -1} (x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \rightarrow -1} h(x) = \lim_{x \rightarrow -1} -h(x) = h(-1)$$

Therefore, h is continuous at $x = -1$. From the above three observations, it can be concluded that h is continuous at all points of the real line. g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function. Therefore, f has no point of discontinuity.

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Q1 Differentiate the functions with respect to x .

$$\sin(x^2 + 5)$$

Answer.

Let $f(x) = \sin(x^2 + 5)$, $u(x) = x^2 + 5$, and $v(t) = \sin t$

$$\text{Then, } (v \circ u)(x) = v(u(x)) = v(x^2 + 5) = \sin(x^2 + 5) = f(x)$$

Thus, f is a composite of two functions.

$$\text{Put } t = u(x) = x^2 + 5$$

Then, we obtain

$$t = u(x) = x^2 + 5$$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \times 2x = 2x \cos(x^2 + 5)$$

Alternate method

$$\begin{aligned} \frac{d}{dx} [\sin(x^2 + 5)] &= \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5) \\ &= \cos(x^2 + 5) \cdot \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(5) \right] \\ &= \cos(x^2 + 5) \cdot [2x + 0] \\ &= 2x \cos(x^2 + 5) \end{aligned}$$

Q2 Differentiate the functions with respect to x.

cos (sin x)

Answer.

Let $f(x) = \cos(\sin x)$, $u(x) = \sin x$, and $v(t) = \cos t$

Then, $(f \circ u)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Thus, f is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

$$\text{By chain rule } \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

Alternate Method

$$\frac{d}{dx}[\cos(\sin x)] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

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Q3 Differentiate the functions with respect to x. sin (ax + b)

Answer.

Let $f(x) = \sin(ax + b)$, $u(x) = ax + b$, and $v(t) = \sin t$

Then, $(f \circ u)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = f(x)$

Thus, f is a composite function of two functions, u and v .

Put $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)$$

Alternate method

$$\begin{aligned} \frac{d}{dx}[\sin(ax + b)] &= \cos(ax + b) \cdot \frac{d}{dx}(ax + b) \\ &= \cos(ax + b) \cdot \left[\frac{d}{dx}(ax) + \frac{d}{dx}(b) \right] \\ &= \cos(ax + b) \cdot (a + 0) \\ &= a \cos(ax + b) \end{aligned}$$

Q4 Differentiate the functions with respect to x. $\sec(\tan(\sqrt{x}))$

Answer.

Let $f(x) = \sec(\tan\sqrt{x})$, $u(x) = \sqrt{x}$, $v(t) = \tan t$, and $w(s) = \sec s$

Then, $(w \circ v \circ u)(x) = w[v(u(x))] = w[v(\sqrt{x})] = w(\tan\sqrt{x}) = \sec(\tan\sqrt{x}) = f(x)$

Thus, f is a composite function of three functions, u , v , and w .

Put $s = v(t) = \tan t$ and $t = u(x) = \sqrt{x}$

$$\begin{aligned}\frac{dw}{ds} &= \frac{d}{ds}(\sec s) = \sec s \sin s = \sec(\tan t) \cdot \tan(\tan t) \quad [s = \tan t] \\ &= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x})\end{aligned}$$

$$\frac{ds}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t = \sec^2 \sqrt{x}$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\begin{aligned}\frac{df}{dx} &= \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} \\ &= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \times \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \sec^2 \sqrt{x} \sec(\tan\sqrt{x}) \tan(\tan\sqrt{x}) \\ &= \frac{\sec^2 \sqrt{x} \sec(\tan\sqrt{x}) \tan(\tan\sqrt{x})}{2\sqrt{x}}\end{aligned}$$

Alternate method

$$\begin{aligned}\frac{d}{dx}[\sec(\tan\sqrt{x})] &= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \frac{d}{dx}(\tan\sqrt{x}) \\ &= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \sec^2(\sqrt{x})}{2\sqrt{x}}\end{aligned}$$

Q5 Differentiate the functions with respect to x

Answer. The given function is $f(x) = \frac{\sin(ax+b)}{\cos(cx+d)} = \frac{g(x)}{h(x)}$ where $g(x) = \sin(ax+b)$ and $h(x) = \cos(cx+d)$

$$\therefore f' = \frac{g'h - gh'}{h^2}$$

Consider $g(x) = \sin(ax+b)$

Let $u(x) = ax+b$, $v(t) = \sin t$

Then, $(v \circ u)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = g(x)$

$\therefore g$ is a composite function of two functions, u and v

Put $t = u(x) = ax+b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a \cos(ax+b)$$

Consider $h(x) = \cos(cx+d)$

Let $p(x) = cx+d$, $q(y) = \cos y$

Then, $(q \circ p)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$

Put $y = p(x) = cx+d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx+d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx+d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx+d) \times c = -c \sin(cx+d)$$

$$\begin{aligned} \therefore f' &= \frac{a \cos(ax+b) \cdot \cos(cx+d) - \sin(ax+b) \{-c \sin(cx+d)\}}{[\cos(cx+d)]^2} \\ &= \frac{a \cos(ax+b)}{\cos(cx+d)} + c \sin(ax+b) \cdot \frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)} \\ &= a \cos(ax+b) \sec(cx+d) + c \sin(ax+b) \tan(cx+d) \sec(cx+d) \end{aligned}$$

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Q6 Differentiate the functions with respect to x . $\cos x^3 \cdot \sin^2(x^5)$

Answer.

The given function is

$$\frac{d}{dx} [\cos x^3 \cdot \sin^2(x^5)] = \sin^2(x^5) \times \frac{d}{dx} (\cos x^3) + \cos x^3 \times \frac{d}{dx} [\sin^2(x^5)]$$

Processing math: 100%

$$\begin{aligned}
&= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2\sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}(x^5) \\
&= -3x^2 \sin x^3 - \sin^2(x^5) + 2\sin x^5 \cos x^3 \cos x^3 \cdot x 5x^4 \\
&= 10x^4 \sin x^5 \cos x^5 \cos x^2 - 3x^2 \sin x^3 \sin^2(x^5)
\end{aligned}$$

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Q7 Differentiate the functions with respect to x. $2\sqrt{\cot(x^2)}$

Answer.

$$\begin{aligned}
&\frac{d}{dx} \left[2\sqrt{\cot(x^2)} \right] \\
&= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx} [\cot(x^2)] \\
&= \sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times -\csc^2(x^2) \times \frac{d}{dx}(x^2) \\
&= -\sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times \frac{1}{\sin^2(x^2)} \times (2x) \\
&= \frac{-2x}{\sqrt{\cos x^2} \sqrt{\sin x^2} \sin x^2} \\
&= \frac{-2\sqrt{2}x}{\sqrt{2\sin x^2 \cos x^2} \sin x^2} \\
&= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}}
\end{aligned}$$

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Q8 Differentiate the functions with respect to x. $\cos(\sqrt{x})$

Answer.

Let $f(x) = \cos(\sqrt{x})$

Also, let $u(x) = \sqrt{x}$

And, $v(t) = \cos t$

Then, $(v \circ u)(x) = v(u(x))$

$$= v(\sqrt{x})$$

$$= \cos \sqrt{x}$$

$$= f(x)$$

Clearly, f is a composite function of two functions, u and v , such that

$$t = u(x) = \sqrt{x}$$

$$\begin{aligned} \text{Then, } \frac{dt}{dx} &= \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

By using chain rule, we obtain

$$\begin{aligned} \frac{df}{dx} &= \frac{dv}{dt} \cdot \frac{dt}{dx} \\ &= -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x}) \\ &= -\frac{\sin(\sqrt{x})}{2\sqrt{x}} \end{aligned}$$

Alternate method

$$\begin{aligned} \frac{d}{dx}[\cos(\sqrt{x})] &= -\sin(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= -\sin(\sqrt{x}) \times \frac{d}{dx}(x^{\frac{1}{2}}) \\ &= -\sin \sqrt{x} \times \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{-\sin \sqrt{x}}{2\sqrt{x}} \end{aligned}$$

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Q9

Prove that the function f given by

$f(x) = |x - 1|$, $x \in \mathbf{R}$ is not differentiable at $x = 1$

Processing math: 100%
Action is $f(x) = |x - 1|$, $x \in \mathbf{R}$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \text{ and } \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} \text{ are finite and equal.}$$

To check the differentiability of the given function at $x = 1$,

consider the left hand limit of f at $x = 1$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^-} \frac{|1+h-1| - |1-1|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad (h < 0 \Rightarrow |h| = -h) \\ &= -1 \end{aligned}$$

Consider the right hand limit of f at $x = 1$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^+} \frac{|1+h-1| - |1-1|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} \quad (h > 0 \Rightarrow |h| = h) \\ &= 1 \end{aligned}$$

since the left and right hand limits of f at $x = 1$ are not equal, f is not differentiable at x

$x = 1$

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Q10

Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$.

Answer.

The given function f is $f(x) = [x]$, $0 < x < 3$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \text{ and } \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} \text{ are finite and equal.}$$

To check the differentiability of the given function at $x = 1$, consider the left hand limit of f at $x = 1$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^-} \frac{[1+h] - [1]}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{1-1}{h} = \lim_{h \rightarrow 0^-} 0 = 0 \end{aligned}$$

Since the left and right hand limits of f are not equal, f is not differentiable at

$x = 1$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{[2+h] - [2]}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{2-2}{h} = \lim_{h \rightarrow 0^+} 0 = 0 \end{aligned}$$

Consider the right hand limit of f at $x = 1$

$$\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{t \rightarrow 2} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{2-2}{h} = \lim_{k \rightarrow 0} 0 = 0$$

since the left and right hand limits of f at $x = 2$ are not equal, f is not differentiable at $x = 2$

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Q1 Find $\frac{dy}{dx}$: in $2x + 3y = \sin x$

Answer.

The given relationship is $2x + 3y = \sin x$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$$

$$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

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Q2 Find $\frac{dy}{dx}$: in $2x + 3y = \sin y$

Answer.

The given relationship is $2x + 3y = \sin y$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y)$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos y \frac{dy}{dx} \quad [\text{By using chain rule}]$$

$$\Rightarrow 2 = (\cos y - 3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

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$$\left(\frac{2}{\cos y - 3} \right)^2 = \cos y$$

Answer.

The given relationship is $ax + by^2 = \cos y$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$$

$$\Rightarrow a + b \frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$ and $\frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx}$

From (1) and (2), we obtain

$$a + b \times 2y \frac{dy}{dx} = -\sin y \frac{dy}{dx}$$

$$\Rightarrow (2by + \sin y) \frac{dy}{dx} = -a$$

$$\therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

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Q4 Find $\frac{dy}{dx}$: in $xy + y^2 = \tan x + y$

Answer.

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow y \cdot 1 + x \cdot \frac{dy}{dx} + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1) \frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

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Q5 Find $\frac{dy}{dx}$: in $x^2 + xy + y^2 = 100$

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to x , we obtain

$$\begin{aligned}\frac{d}{dx}(x^2 + xy + y^2) &= \frac{d}{dx}(100) \\ \Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= 0 \\ \Rightarrow 2x + \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} &= 0 \\ \Rightarrow 2x + y \cdot 1 + x \cdot \frac{dy}{dx} + 2y \frac{dy}{dx} &= 0 \\ \Rightarrow 2x + y + (x + 2y) \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx} &= -\frac{2x+y}{x+2y}\end{aligned}$$

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Q6 Find $\frac{dy}{dx}$: in $x^3 + x^2y + xy^2 + y^3 = 81$

Answer.

The given relationship is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x , we obtain

$$\begin{aligned}\frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) &= \frac{d}{dx}(81) \\ \Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) &= 0 \\ \Rightarrow 3x^2 + \left[y \frac{d}{dx}(x^2) + x^2 \frac{dy}{dx} \right] + \left[y^2 \frac{d}{dx}(x) + x \frac{d}{dx}(y^2) \right] + 3y^2 \frac{dy}{dx} &= 0 \\ \Rightarrow 3x^2 + \left[y \cdot 2x + x^2 \frac{dy}{dx} \right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^2 \frac{dy}{dx} &= 0 \\ \Rightarrow (x^2 + 2xy + 3y^2) \frac{dy}{dx} + (3x^2 + 2xy + y^2) &= 0 \\ \therefore \frac{dy}{dx} &= \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}\end{aligned}$$

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Q7 Find $\frac{dy}{dx}$: in $\sin^2y + \cos xy = \pi$

Processing math: 100%

Answer.

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin^2 y) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx}$$

$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy \left[y \frac{d}{dx}(x) + x \frac{dy}{dx} \right]$$

$$= -\sin xy \left[y \cdot 1 + x \frac{dy}{dx} \right] = -y \sin xy - x \sin xy \frac{dy}{dx}$$

From above results we get

$$2\sin y \cos y \frac{dy}{dx} - y \sin xy - x \sin xy \frac{dy}{dx} = 0$$

$$\Rightarrow (2\sin y \cos y - x \sin xy) \frac{dy}{dx} = y \sin xy$$

$$\Rightarrow (\sin 2y - x \sin xy) \frac{dy}{dx} = y \sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}$$

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Q8 Find $\frac{dy}{dx}$: in $\sin^2 x + \cos^2 y = 1$

Answer.

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(\sin^2 x + \cos^2 y) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 x) + \frac{d}{dx}(\cos^2 y) = 0$$

$$\Rightarrow 2\sin x \cdot \frac{d}{dx}(\sin x) + 2\cos y \cdot \frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow 2\sin x \cos x + 2\cos y (-\sin y) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x \sin 2y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Processing math: 100%

Q9 Find $\frac{dy}{dx}$: in $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Answer. The given relationship is

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\frac{2x}{1+x^2}, \text{ is of the form of } v$$

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{\left(1+x^2\right) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$= \frac{\left(1+x^2\right) \cdot 2 - 2x \cdot [0+2x]}{\left(1+x^2\right)^2} = \frac{2+2x^2-4x^2}{\left(1+x^2\right)^2} = \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{\left(1+x^2\right)^2 - 4x^2}{\left(1+x^2\right)^2}}$$

$$= \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \frac{1-x^2}{1+x^2}$$

$$\frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

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Q10 Find $\frac{dy}{dx}$: in $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Answer. The given relationship is

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x-x^3}{1-3x^2}$$

It is know that $\tan y = \frac{3\tan \frac{y}{3} - \tan^3 \frac{y}{3}}{1 - 3\tan^2 \frac{y}{3}}$

on comparing above two results

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan \frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2 \frac{y}{3} \cdot \frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2 \frac{y}{3} \cdot \frac{1}{3} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2 \frac{y}{3}} = \frac{3}{1 + \tan^2 \frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$$

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Q11 Find $\frac{dy}{dx}$: in $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0 < x < 1$

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Answer. The given relationship is,

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1 - \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan \frac{y}{2} = x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2 \frac{y}{2} \cdot \frac{d}{dx} \left(\frac{y}{2} \right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2 \frac{y}{2} \times \frac{1}{2} \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1 + \tan^2 \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

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Q12 Find $\frac{dy}{dx}$: in $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer. The given relationship is

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx} \left(\frac{1-x^2}{1+x^2} \right)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\begin{aligned} \cos y &= \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{1-x^2}{1+x^2} \right)^2} \\ &= \sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{64x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2} \end{aligned}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2} \frac{dy}{dx}$$

$$\frac{d}{dx} \left(\frac{1-x^2}{1+x^2} \right) = \frac{(1+x^2) \cdot (1-x^2)' - (1-x^2) \cdot (1+x^2)'}{(1+x^2)^2}$$

$$= \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}$$

From above results

$$= \frac{-4x}{(1+x^2)^2}$$

$$\frac{2x}{1+x^2} \frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Alternate method

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\sin y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow (1+x^2)\sin y = 1-x^2$$

$$\Rightarrow (1+x^2)\sin y = 1-x^2$$

$$\Rightarrow (1+\sin y)x^2 = 1-x^2$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos \frac{y}{2} - \sin \frac{y}{2}\right)^2}{\left(\cos \frac{y}{2} + \sin \frac{y}{2}\right)^2}$$

$$\Rightarrow x = \frac{\cos \frac{y}{2} - \sin \frac{y}{2}}{\cos \frac{y}{2} + \sin \frac{y}{2}}$$

$$\Rightarrow x = \frac{1 - \tan \frac{y}{2}}{1 + \tan \frac{y}{2}}$$

$$\Rightarrow x = \frac{1 - \tan \frac{y}{2}}{1 + \tan \frac{y}{2}}$$

$$\Rightarrow x = \tan\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx} \cdot \left[\tan\left(\frac{\pi}{4} - \frac{y}{2}\right) \right]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

$$\Rightarrow 1 = 1 = \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right) \right] \cdot \left(-\frac{1}{2} \frac{dy}{dx} \right)$$

$$\Rightarrow 1 = (1+x^2) \left(-\frac{1}{2} \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

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Q13 Find $\frac{dy}{dx}$: in $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1 < x < 1$

Answer. The given relationship is

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\cos y) = \frac{d}{dx} \cdot \left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = \frac{(1+x^2) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y} \frac{dy}{dx} = \frac{(1+x^2) \times 2 - 2x \cdot 2x}{(1+x^2)^2}$$

$$\Rightarrow \left[\sqrt{1 - \left(\frac{2x}{1+x^2}\right)^2} \right] \frac{dy}{dx} = - \left[\frac{2(1-x^2)}{(1+x^2)^2} \right]$$

$$\Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} \frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2} \cdot \frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Processing math: 100% $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Answer.

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 2 \left[x \frac{d}{dx} \left(\sqrt{1-x^2} \right) + \sqrt{1-x^2} \frac{dx}{dx} \right]$$

$$\Rightarrow \sqrt{1-\sin^2 y} \frac{dy}{dx} = 2 \left[\frac{x}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2} \frac{dy}{dx} = 2 \left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}} \right]$$

$$\Rightarrow \sqrt{1-4x^2(1-x^2)} \frac{dy}{dx} = 2 \left[\frac{1-2x^2}{\sqrt{1-x^2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

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Q15 Find $\frac{dy}{dx}$: in $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$, $0 < x < \frac{1}{\sqrt{2}}$

Answer. The given relationship is

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx} \left(\cos \frac{y}{2} \right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx} \left(\frac{y}{2} \right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

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Q1 Differentiate the following w.r.t. x: $\frac{e^x}{\sin x}$

Answer. Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx} (e^x) - e^x \frac{d}{dx} (\sin x)}{\sin^2 x}$$

$$= \frac{\sin x \cdot (e^x) - e^x \cdot (\cos x)}{\sin^2 x}$$

$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbb{Z}$$

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Q2 Differentiate the following w.r.t. x:

Processing math: 100%

Answer. Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(e^{\sin^{-1} x} \right) \\ \Rightarrow \frac{dy}{dx} &= e^{\sin^{-1} x} \cdot \frac{d}{dx} \left(\sin^{-1} x \right) \\ &= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} \\ &= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \\ \therefore \frac{dy}{dx} &= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1, 1)\end{aligned}$$

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Q3 Differentiate the following w.r.t. x: e^{x^3}

Answer.

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx} \left(e^{x^3} \right) = e^{x^3} \cdot \frac{d}{dx} \left(x^3 \right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

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Q4 Differentiate the following w.r.t. x: $\sin \left(\tan^{-1} e^{-x} \right)$

Answer.

Let $y = \sin \left(\tan^{-1} e^{-x} \right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx} \left[\sin \left(\tan^{-1} e^{-x} \right) \right]$$

$$\begin{aligned}
&= \cos\left(\tan^{-1}e^{-x}\right) \cdot \frac{1}{1 + \left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right) \\
&= \frac{\cos\left(\tan^{-1}e^{-x}\right)}{1 + e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x) \\
&= \frac{e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1 + e^{-2x}} \times (-1) \\
&= \frac{-e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1 + e^{-2x}}
\end{aligned}$$

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Q5 Differentiate the following w.r.t. x: $\log(\csc x)$

Answer. $y = \log(\csc x)$

By using the chain rule, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left[\log(\csc x) \right] \\
&= \frac{1}{\csc x} \cdot \frac{d}{dx} (\csc x) \\
&= \frac{1}{\csc x} \cdot (-\sin x) \cdot \frac{d}{dx} (x) \\
&= \frac{-\sin x}{\csc x} \cdot 1 \\
&= -\sin^2 x, \sin x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{N}
\end{aligned}$$

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Q6 Differentiate the following w.r.t. X: $e^x + e^{x^2} + \dots + e^{x^5}$

Answer.

$$\begin{aligned}
&\frac{d}{dx} \left(e^x + e^{x^2} + \dots + e^{x^5} \right) \\
&= \frac{d}{dx} (e^x) + \frac{d}{dx} (e^{x^2}) + \frac{d}{dx} (e^{x^3}) + \frac{d}{dx} (e^{x^4}) + \frac{d}{dx} (e^{x^5})
\end{aligned}$$

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$$\begin{aligned}
&= e^x + \left[e^{x^2} \times \frac{d}{dx}(x^2) \right] + \left[e^{x^3} \cdot \frac{d}{dx}(x^3) \right] + \left[e^{x^4} \cdot \frac{d}{dx}(x^4) \right] + \left[e^{x^5} \cdot \frac{d}{dx}(x^5) \right] \\
&= e^x + \left(e^{x^2} \times 2x \right) + \left(e^{x^3} \times 3x^2 \right) + \left(e^{x^4} \times 4x^3 \right) + \left(e^{x^5} \times 5x^4 \right) \\
&= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}
\end{aligned}$$

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Q7 Differentiate the following w.r.t. X: $\sqrt{e^{\sqrt{x}}}$, $x > 0$

Answer. $y = \sqrt{e^{\sqrt{x}}}$

$$y^2 = e^{\sqrt{x}}$$

By differentiating this relationship with respect to x, we obtain

$$y^2 = e^{\sqrt{x}}$$

$$\Rightarrow 2y \frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow 2y \frac{dy}{dx} = e^{\sqrt{x}} \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$$

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Q8 Differentiate the following w.r.t. X: $\log(\log x)$, $x > 1$

Answer.

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\log x)]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \cdot \frac{1}{x}$$

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$x \log x$

Q9 Differentiate the following w.r.t. X: $\frac{\cos x}{\log x}, x > 0$

Answer.

$$y = \frac{\cos x}{\log x}$$

By using the quotient rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2} \\ &= \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2} \\ &= \frac{-\left[x \log x - \cos x \times \frac{1}{x} \right]}{(\log x)^2} \end{aligned}$$

Q10 Differentiate the following w.r.t. X: $\cos(\log x + e^x), x > 0$

Answer. $y = \cos(\log x + e^x)$

$$\begin{aligned} \frac{dy}{dx} &= -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x) \\ &= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x) \right] \\ &= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x \right) \\ &= -\left(\frac{1}{x} + e^x \right) \sin(\log x + e^x), x > 0 \end{aligned}$$

Q1 Differentiate the function with respect to x.

$$\cos x \cdot \cos 2x \cdot \cos 3x$$

Answer. $y = \cos x \cdot \cos 2x \cdot \cos 3x$

Taking logarithm on both the sides, we obtain

Processing math: 100%

$$\log y = \log(\cos x \cdot \cos 2x \cdot \cos 3x)$$

$$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \frac{1}{\cos 2x} \cdot \frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x} \cdot \frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \cdot \frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x} \cdot \frac{d}{dx}(3x) \right]$$

$$\therefore \frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x [\tan x + 2\tan 2x + 3\tan 3x]$$

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Q2 Differentiate the function with respect to x.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer. Let

$$y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2} \log \left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right]$$

$$\Rightarrow \log y = \frac{1}{2} [\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}]$$

$$\Rightarrow \log y = \frac{1}{2} [\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x-1} \cdot \frac{d}{dx}(x-1) + \frac{1}{x-2} \cdot \frac{d}{dx}(x-2) - \frac{1}{x-3} \cdot \frac{d}{dx}(x-3) - \frac{1}{x-4} \cdot \frac{d}{dx}(x-4) - \frac{1}{x-5} \cdot \frac{d}{dx}(x-5) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2} \left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]$$

Q3 Differentiate the function with respect to x.

$$(\log x)^{\cos x}$$

Answer.

$$\text{Let } y = (\log x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = -\sin x \log(\log x) + \cos x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x} \right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin x \log(\log x) \right]$$

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Q4 Differentiate the function with respect to x.

$$x^x - 2^{\ln x}$$

Answer.

$$\text{Let } y = x^x - 2^{\ln x}$$

$$\text{Also, let } x^x = u \text{ and } 2^{\ln x} = v$$

$$\therefore y = u - v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$u = x^x$$

Taking logarithm on both the sides, we obtain

$$\log u = x \log x$$

$$\frac{1}{u} \frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x) \right]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \times \log x + x \times \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^x (\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x (1 + \log x)$$

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Taking logarithm on both the sides with respect to x, we obtain

$$\log v = \sin x \cdot \log 2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$$

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Q5 Differentiate the function with respect to x.

$$(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$$

Answer.

$$\text{Let } y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) + 3 \cdot \frac{1}{x+4} \cdot \frac{d}{dx}(x+4) + 4 \cdot \frac{1}{x+5} \cdot \frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3 \cdot \left[2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12) \right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2 + 70x + 133)$$

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Q6 Differentiate the function with respect to x.

$$\left(x + \frac{1}{x}\right)^x + x \left(1 + \frac{1}{x}\right)$$

$$\text{Let } y = \left(x + \frac{1}{x}\right)^x + x \left(1 + \frac{1}{x}\right)$$

$$\text{Also, let } u = \left(x + \frac{1}{x}\right)^x \text{ and } v = x \left(1 + \frac{1}{x}\right)$$

$$\begin{aligned} \therefore y &= u + v \\ \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} + \frac{dv}{dx} \end{aligned}$$

$$\text{Then } u = \left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log u = \log \left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x \log \left(x + \frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x) \times \log \left(x + \frac{1}{x}\right) + x \times \frac{d}{dx} \left[\log \left(x + \frac{1}{x}\right) \right]$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = 1 \times \log \left(x + \frac{1}{x}\right) + x \times \frac{1}{\left(x + \frac{1}{x}\right)} \cdot \frac{d}{dx} \left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\log \left(x + \frac{1}{x}\right) + \frac{\left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)} \right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\log \left(x + \frac{1}{x}\right) + \frac{x^2 - 1}{x^2 + 1} \right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x}\right) \right]$$

$$v = x \left(1 + \frac{1}{x}\right)$$

$$y = x \left(1 + \frac{1}{x} \right)$$

$$\Rightarrow \log v = \log \left[x \left(1 + \frac{1}{x} \right) \right]$$

$$\Rightarrow \log v = \left(1 + \frac{1}{x} \right) \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \left[\frac{d}{dx} \left(1 + \frac{1}{x} \right) \right] \times \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{d}{dx} \log x$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left(-\frac{1}{x^2} \right) \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v \left[\frac{-\log x + x + 1}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = x \left(1 + \frac{1}{x} \right) \left(\frac{x + 1 - \log x}{x^2} \right)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = \left(x + \frac{1}{x} \right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x} \right) \right] + x \left(1 + \frac{1}{x} \right) \left(\frac{x + 1 - \log x}{x^2} \right)$$

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Q7 Differentiate the function with respect to x .

$$(\log x)^x + x^{\log x}$$

Answer.

$$\text{Let } y = (\log x)^x + x^{\log x}$$

$$\text{Also, let } u = (\log x)^x \text{ and } v = x^{\log x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (\log x)^x$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \log u = \log \left[(\log x)^x \right]$$

$$\Rightarrow \log u = x \log(\log x)$$

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx}(x) \times \log(\log x) + x \cdot \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \times \log(\log x) + x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{x}{\log x} \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\frac{\log(\log x) \cdot \log x + 1}{\log x} \right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1} [1 + \log x \cdot \log(\log x)]$$

$$y = x^{\log x}$$

$$\Rightarrow \log v = \log \left(x^{\log x} \right)$$

$$\Rightarrow \log v = \log \left(x^{\log x} \right) = (\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx} \left[(\log x)^2 \right]$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = 2(\log x) \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2v(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1} \cdot \log x$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{n-1} [1 + \log x \cdot \log(\log x)] + 2x^{\log x - 1} \cdot \log x$$

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Q8 Differentiate the function with respect to x.

$$(\sin x)^x + \sin^{-1} \sqrt{x}$$

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Answer.

$$\text{Let } y = (\sin x)^x + \sin^{-1}\sqrt{x}$$

$$\text{Also, let } u = (\sin x)^x \text{ and } v = \sin^{-1}\sqrt{x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (\sin x)^x$$

$$\Rightarrow \log u = \log(\sin x)^x$$

$$\Rightarrow \log u = x \log(\sin x)$$

Differentiating both sides with respect to x , we obtain

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(x) \times \log(\sin x) + x \times \frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \cdot \log(\sin x) + x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\log(\sin x) + \frac{x}{\sin x} \cdot \cos x \right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^T (x \cot x + \log \sin x)$$

$$v = \sin^{-1}\sqrt{x}$$

Differentiating both sides with respect to x , we obtain

$$\Rightarrow \frac{dv}{dx} = (\sin x)^x \left[\log(\sin x) + \frac{x}{\sin x} \cdot \cos x \right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^T (x \cot x + \log \sin x)$$

$$v = \sin^{-1}\sqrt{x}$$

Differentiating both sides with respect to x , we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)' (x \cot x + \log \sin x)$$

$$v = \sin^{-1}\sqrt{x}$$

Differentiating both sides with respect to x , we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}}$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}$$

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Q9 Differentiate the function with respect to x.

$$x^{\sin x} + (\sin x)^{\cos x}$$

Answer.

$$\text{Let } y = x^{\sin x} + (\sin x)^{\cos x}$$

$$\text{Also, let } u = x^{\sin x} \text{ and } v = (\sin x)^{\cos x}$$

$$\therefore y = u + v$$

$$u = x^{\sin x}$$

$$\Rightarrow \log u = \log(x^{\sin x})$$

$$\Rightarrow \log u = \sin x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx}(\sin x) \cdot \log x + \sin x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\cos x \log x + \sin x \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right]$$

$$v = (\sin x)^{\cos x}$$

$$\Rightarrow \log v = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx}(\cos x) \times \log(\sin x) + \cos x \times \frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dv}{dx} = v \left[-\sin x \cdot \log(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} \left[-\sin x \log \sin x + \frac{\cos x}{\sin x} \cos x \right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} [-\sin x \log \sin x + \cot x \cos x]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} [\cot x \cos x - \sin x \log \sin x]$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x} \left(\cos x \log x + \frac{\sin x}{x} \right) + (\sin x)^{\cos x} [\cos x \cot x - \sin x \log \sin x]$$

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Q10 Differentiate the function with respect to x.

$$x^{x \sin x} + \frac{x^2 + 1}{x^2 - 1}$$

Answer.

$$\text{Let } y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$$

$$\text{Also, let } u = x^{\sin x} \text{ and } v = \frac{x^2 + 1}{x^2 - 1}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = x^{\log x}$$

$$\Rightarrow \log u = \log(x^{\sin x})$$

$$\Rightarrow \log u = x \cos x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx}(x) \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \cdot \cos x \cdot \log x + x \cdot (-\sin x) \log x + x \cos x \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^{\cos x} (\cos x \log x - x \sin x \log x + \cos x)$$

$$v = \frac{x^2 + 1}{x^2 - 1}$$

$$\Rightarrow \log v = \log(x^2 + 1) - \log(x^2 - 1)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \frac{dv}{dx} = \frac{2x}{x^2+1} - \frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx} = v \left[\frac{2x(x^2-1) - 2x(x^2+1)}{(x^2+1)(x^2-1)} \right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2+1}{x^2-1} \times \left[\frac{-4x}{(x^2+1)(x^2-1)} \right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2-1)^2}$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x} [\cos x (1 + \log x) - x \sin x \log x] - \frac{4x}{(x^2-1)^2}$$

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Q11 Differentiate the function with respect to x.

$$(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

Answer.

$$\text{Let } y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

$$\text{Also, let } u = (x \cos x)^x \text{ and } v = (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x \cos x)^x$$

$$\Rightarrow \log u = \log (x \cos x)^x$$

$$\Rightarrow \log u = x \log (x \cos x)$$

$$\Rightarrow \log u = x [\log x + \log \cos x]$$

$$\Rightarrow \log u = x \log x + x \log \cos x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x \log x) + \frac{d}{dx} (x \log \cos x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\left\{ \log x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log x) \right\} + \left\{ \log \cos x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log \cos x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[\left(\log x \cdot 1 + x \cdot \frac{1}{x} \right) + \left\{ \log \cos x \cdot 1 + x \cdot \frac{1}{\cos x} \cdot \frac{d}{dx} (\cos x) \right\} \right]$$

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$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[(\log x + 1) + \left\{ \log \cos x + \frac{x}{\cos x} \cdot (-\sin x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x [(1 + \log x) + (\log \cos x - x \tan x)]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x [1 - x \tan x + (\log x + \log \cos x)]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x [1 - x \tan x + \log(x \cos x)]$$

$$v = (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log(x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log(x \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} (\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$$

$$\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{x} \log x \right) + \frac{d}{dx} \left[\frac{1}{x} \log(\sin x) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[\log x \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} (\log x) \right] + \left[\log(\sin x) \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} \{ \log(\sin x) \} \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[\log x \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{x} \right] + \left[\log(\sin x) \cdot \left(\frac{1}{x^2} \right) + \frac{1}{x^2} + \frac{1}{x \sin x} \cdot \frac{d}{dx} (\sin x) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2} (1 - \log x) + \left[\frac{\log(\sin x)}{x^2} + \frac{1}{x \sin x} + \frac{1}{x \sin x} \right]$$

$$\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x - \log(\sin x) + x \cot x}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log(x \sin x) + x \cot x}{x^2} \right]$$

From (1), (2), and (3), we obtain

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x \cos x)^x [1 - x \tan x + \log(x \cos x)] + (x \sin x)^{\frac{1}{x}} \left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right]$$

Q12 Find $\frac{dy}{dx}$ of function.

$$x^y + y^x = 1$$

Answer.

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$u = x^y$$

$$\Rightarrow \log u = \log(x^y)$$

$$\Rightarrow \log u = y \log x$$

$$\frac{1}{u} \frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right)$$

$$v = y^x$$

$$\Rightarrow \log v = \log(y^x)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v \left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dv}{dx} = y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right)$$

From (1), (2), and (3), we obtain

$$x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \left(x^y \log x + x y^{x-1} \right) \frac{dy}{dx} = - \left(y x^{y-1} + y^x \log y \right)$$

$$\therefore \frac{dy}{dx} = - \frac{y x^{y-1} + y^x \log y}{x^y \log x + x y^{x-1}}$$

Q13 Find $\frac{dy}{dx}$ of function.

$$y^x = x^y$$

Answer.

The given function is $x^y + y^x = 1$

Let $x' = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$u = x^r$$

$$\Rightarrow \log u = \log(x^x)$$

$$\Rightarrow \log u = y \log x$$

$$\frac{1}{u} \frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right)$$

$$v = y^k$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right)$$

From (1), (2), and (3), we obtain

$$x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \left(x^y \log x + x y^{x-1} \right) \frac{dy}{dx} = - \left(y x^{y-1} + y^x \log y \right)$$

$$\therefore \frac{dy}{dx} = - \frac{y x^{y-1} + y^x \log y}{x^y \log x + x y^{x-1}}$$

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Q14 Find $\frac{dy}{dx}$ of function.

$$(\cos x)^x = (\cos y)^x$$

Answer.

Processing math: 100%

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides, we obtain

$$\begin{aligned} \log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) &= \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y) \\ \Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) &= \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y) \\ \Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) &= \log \cos y + \frac{x}{\cos y} (-\sin y) \cdot \frac{dy}{dx} \\ \Rightarrow \log \cos x \frac{dy}{dx} - y \tan x &= \log \cos y - x \tan y \frac{dy}{dx} \\ \Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} &= y \tan x + \log \cos y \\ \therefore \frac{dy}{dx} &= \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x} \\ \therefore \frac{dy}{dx} &= \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x} \end{aligned}$$

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Q15 Find $\frac{dy}{dx}$ of function.

$$xy = e^{(x-y)}$$

Answer.

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned} \log(xy) &= \log(e^{x-y}) \\ \Rightarrow \log x + \log y &= (x-y) \log e \\ \Rightarrow \log x + \log y &= (x-y) \times 1 \\ \Rightarrow \log x + \log y &= (x-y) \times 1 \\ \Rightarrow \log x + \log y &= x-y \\ \frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) &= \frac{d}{dx}(x) - \frac{dy}{dx} \\ \Rightarrow \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} &= 1 - \frac{dy}{dx} \\ \Rightarrow \left(1 + \frac{1}{y}\right) \frac{dy}{dx} &= 1 - \frac{1}{x} \\ \Rightarrow \left(\frac{y+1}{y}\right) \frac{dy}{dx} &= \frac{x-1}{x} \end{aligned}$$

$$\frac{dv}{v} = \frac{v(x-1)}{v(x-1)}$$

Processing math: 100%

Q16 Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$

Answer. The given relationship is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ Taking logarithm on both the sides, we obtain

$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$ Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{f(x)} \cdot \frac{d}{dx}[f(x)] &= \frac{d}{dx} \log(1+x) + \frac{d}{dx} \log(1+x^2) + \frac{d}{dx} \log(1+x^4) + \frac{d}{dx} \log(1+x^8) \\ \Rightarrow \frac{1}{f(x)} \cdot f'(x) &= \frac{1}{1+x} \cdot \frac{d}{dx}(1+x) + \frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2) + \frac{1}{1+x^4} \cdot \frac{d}{dx}(1+x^4) + \frac{1}{1+x^8} \cdot \frac{d}{dx}(1+x^8) \\ \Rightarrow f'(x) &= f(x) \left[\frac{1}{1+x} + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^8} \cdot 8x^7 \right] \end{aligned}$$

$$\therefore f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} \right]$$

$$\text{Hence, } f'(1) = (1+1)(1+1^2)(1+1^4)(1+1^8) \left[\frac{1}{1+1} + \frac{2 \times 1}{1+1^2} + \frac{4 \times 1^3}{1+1^4} + \frac{8 \times 1^7}{1+1^8} \right]$$

$$= 2 \times 2 \times 2 \times 2 \left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2} \right]$$

$$= 16 \times \left(\frac{1+2+4+8}{2} \right)$$

$$= 16 \times \frac{15}{2} = 120$$

Q17 Differentiate $(x^4 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below

- By using product rule.
- By expanding the product to obtain a single polynomial.
- By logarithmic differentiation. Do they all give the same answer?

$$\frac{d}{dx} (x^4 - 5x + 8)(x^3 + 7x + 9)$$

(i)

Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

$$\therefore y = uv$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8) \cdot (x^3 + 7x + 9) + (x^2 - 5x + 8) \cdot \frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7) - 5x(3x^2 + 7) + 8(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii)

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

$$= x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)$$

$$= \frac{d}{dx}(x^5) - 5 \frac{d}{dx}(x^4) + 15 \frac{d}{dx}(x^3) - 26 \frac{d}{dx}(x^2) + 11 \frac{d}{dx}(x) + \frac{d}{dx}(72)$$

$$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii)

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
\frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} \log(x^2 - 5x + 8) + \frac{d}{dx} \log(x^3 + 7x + 9) \\
\Rightarrow \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x^2 - 5x + 8} \cdot \frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9} \cdot \frac{d}{dx}(x^3 + 7x + 9) \\
\Rightarrow \frac{dy}{dx} &= y \left[\frac{1}{x^2 - 5x + 8} \times (2x - 5) + \frac{1}{x^3 + 7x + 9} \times (3x^2 + 7) \right] \\
\Rightarrow \frac{dy}{dx} &= (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9} \right] \\
\Rightarrow \frac{dy}{dx} &= 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + 3x^2(x^2 - 5x + 8) + 7(x^2 - 5x + 8) \\
\Rightarrow \frac{dy}{dx} &= (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 - 15x^4 + 24x^2) + (7x^2 - 35x + 56) \\
\Rightarrow \frac{dy}{dx} &= 5x^4 - 20x^3 + 45x^2 - 52x + 4
\end{aligned}$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same .

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Q18 If u, v and w are functions of x , then show that

$$\frac{d}{dx}(u, v, w) = \frac{du}{dx}vw + u \cdot \frac{dv}{dx}w + u, v, \frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer.

Let $y = u, v, w = u \cdot (v, w)$

By applying product rule, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{du}{dx} \cdot (v, w) + u \cdot \frac{d}{dx}(v \cdot w) \\
\Rightarrow \frac{dy}{dx} &= \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}
\end{aligned}$$

By taking logarithm on both sides of the equation $y = u \cdot v \cdot w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w) \\
\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \\
\Rightarrow \frac{dy}{dx} &= y \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right)
\end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = u, v, w \cdot \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

Page : 179 , Block Name : Exercise 5.5

Q1 If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x = 2at^2, y = at^4$$

Answer.

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a \cdot \frac{d}{dt}(t^2) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}(at^4) = a \cdot \frac{d}{dt}(t^4) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{4at^3}{4at} = t^2$$

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Q2 If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x = a \cos \theta, y = b \cos \theta$$

Answer.

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) = a(-\sin \theta) = -a \sin \theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b \cos \theta) = b(-\sin \theta) = -b \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)} = \frac{-b \sin \theta}{-a \sin \theta} = \frac{b}{a}$$

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Processing math: 100%

Q3 If x and y are connected parametrically by the equation, without eliminating the parameter,

find $\frac{dy}{dx}$

$$x = \sin t, y = \cos 2t$$

Answer.

The given equations are $x = \sin t$ and $y = \cos 2t$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$$

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Q4 If x and y are connected parametrically by the equation, without eliminating the parameter,

find $\frac{dy}{dx}$

$$x = 4t, y = \frac{4}{y}$$

Answer. The given equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

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Q5 If x and y are connected parametrically by the equation, without eliminating the parameter,

find $\frac{dy}{dx}$

$$x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$$

Answer.

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) = \frac{d}{d\theta}(\cos 2\theta)$$

$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

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Q6 If x and y are connected parametrically by the equation, without eliminating the parameter,

find $\frac{dy}{dx}$

$$x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$$

Answer.

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a \left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta) \right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a \left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta) \right] = a[0 + (-\sin\theta)] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{dx} = \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

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Q7 If x and y are connected parametrically by the equation, without eliminating the parameter,

find $\frac{dy}{dx}$

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer. The given equation are $\frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\begin{aligned}
\text{Then, } \frac{dx}{dt} &= \frac{d}{dt} \left[\frac{\sin^3 t}{\sqrt{\cos 2t}} \right] \\
&= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt} (\sin^3 t) - \sin^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t} \\
&= \frac{\sqrt{\cos 2t} \cdot 3\sin^2 t \cdot \frac{d}{dt} (\sin t) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt} (\cos 2t)}{\cos 2t} \\
&= \frac{3\sqrt{\cos 2t} \cdot \sin^2 t \cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}} \cdot (-2\sin 2t)}{\cos 2t} \\
&= \frac{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}{\cos 2t \sqrt{\cos 2t}} \\
&= \frac{\sqrt{\cos 2t} \cdot 3\cos^2 t \cdot \frac{d}{dr} (\cos t) - \cos^3 t \cdot \frac{d}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt} (\cos 2t)}{\cos 2t} \\
&= \frac{3\sqrt{\cos 2t} \cos^2 t (-\sin t) - \cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} \cdot (-2\sin 2t)}{\cos 2t} \\
&= \frac{-3\cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}} \\
\therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{-3\cos 2t \cdot \cos^2 t \sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t} \\
&= \frac{-3\cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t (2\sin t \cos t)}{3\cos 2t \sin^2 t \cos t + \sin^3 t (2\sin t \cos t)} \\
&= \frac{\sin t \cos t \left[-3\cos 2t \cdot \cos t + 2\cos^3 t \right]}{\sin t \cos t \left[3\cos 2t \sin t + 2\sin^3 t \right]} \\
&= \frac{\left[-3(2\cos^2 t - 1)\cos t + 2\cos^3 t \right]}{\left[3(1 - 2\sin^2 t)\sin t + 2\sin^3 t \right]} \\
&= \frac{-4\cos^2 t + 3\cos t}{3\sin t - 4\sin^3 t} \\
&= \frac{-\cos 3t}{\sin 3t} \\
&= -\cot 3t
\end{aligned}$$

find $\frac{dy}{dx} x = a \left(\cos t + \log \tan \frac{t}{2} \right), y = a \sin t$

Answer. The given equations are $x = a \left(\cos t + \log \tan \frac{t}{2} \right)$ and $y = a \sin t$

Then, $\frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dr} \left(\log \tan \frac{t}{2} \right) \right]$

$$= a \left[-\sin t + \frac{1}{\tan \frac{t}{2}}, \frac{d}{dt} \left(\tan \frac{t}{2} \right) \right]$$

$$= a \left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt} \left(\frac{t}{2} \right) \right]$$

$$= a \left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right]$$

$$= a \left(-\sin t + \frac{1}{\sin t} \right)$$

$$= a \left(\frac{-\sin^2 t + 1}{\sin t} \right)$$

$$= a \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \frac{d}{dt}(\sin t) = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{a \cos t}{\left(a \frac{\cos^2 t}{\sin t} \right)} = \frac{\sin t}{\cos t} = \tan t$$

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Q9 If x and y are connected parametrically by the equation, without eliminating the parameter, find

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Answer.

The given equations are $x = a \sec \theta$ and $y = b \tan \theta$

$$\text{Then, } \frac{dx}{d\theta} = a \cdot \frac{d}{d\theta}(\sec \theta) = a \sec \theta \tan \theta$$

$$\frac{dy}{d\theta} = b \cdot \frac{d}{d\theta}(\tan \theta) = b \sec^2 \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b}{a} \sec \theta \cot \theta = \frac{b \cos \theta}{a \cos \theta \sin \theta} = \frac{b}{a} \times \frac{1}{\sin \theta} = \frac{b}{a} \csc \theta$$

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Q10 If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$$

Answer.

The given equations are $(x = a(\cos \theta + \theta \sin \theta))$ and $y = a(\sin \theta - \theta \cos \theta)$

$$\text{Then, } \frac{dx}{d\theta} = a \left[\frac{d}{d\theta} \cos \theta + \frac{d}{d\theta}(\theta \sin \theta) \right] = a \left[-\sin \theta + \theta \frac{d}{d\theta}(\sin \theta) + \sin \theta \frac{d}{d\theta}(\theta) \right]$$

$$= a[-\sin \theta + \theta \cos \theta + \sin \theta] = a\theta \cos \theta$$

$$\frac{dy}{d\theta} = a \left[\frac{d}{d\theta}(\sin \theta) - \frac{d}{d\theta}(\theta \cos \theta) \right] = a \left[\cos \theta - \left\{ \theta \frac{d}{d\theta}(\cos \theta) + \cos \theta \cdot \frac{d}{d\theta}(\theta) \right\} \right]$$

$$= a[\cos \theta + \theta \sin \theta - \cos \theta]$$

$$= a\theta \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$$

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Q11 $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} y}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer. The given equations are $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

$$x = \sqrt{a^{\sin^{-1} t}} \text{ and } y = \sqrt{a^{\cos^{-1} t}}$$

$$\Rightarrow x = \left(a^{\sin^{-1} t}\right)^{\frac{1}{2}} \text{ and } y = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}}$$

$$\Rightarrow x = a^{\frac{1}{2}\sin^{-1} t} \text{ and } y = a^{\frac{1}{2}\cos^{-1} t}$$

$$\text{Consider } x = a^{\frac{1}{2}\sin^{-1} t}$$

Taking logarithm on both the sides, we obtain

$$\log x = \frac{1}{2}\sin^{-1} t \log a$$

$$\therefore \frac{1}{x} \cdot \frac{dx}{dt} = \frac{1}{2} \log a \cdot \frac{d}{dt}(\sin^{-1} t)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2} \log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x \log a}{2\sqrt{1-t^2}}$$

$$\log x = \frac{1}{2}\sin^{-1} t \log a$$

$$\therefore \frac{1}{x} \cdot \frac{dx}{dt} = \frac{1}{2} \log a \cdot \frac{d}{dt}(\sin^{-1} t)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2} \log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x \log a}{2\sqrt{1-t^2}}$$

$$\log y = \frac{1}{2}\cos^{-1} t \log a$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \log a \cdot \frac{d}{dt}(\cos^{-1} t)$$

$$\Rightarrow \frac{dy}{dt} = \frac{y \log a}{2} \cdot \left(\frac{-1}{\sqrt{1-t^2}} \right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{-y \log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{\frac{dy}{dx}}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y \log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x \log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}$$

Hence proved

Q1 Find the second order derivatives of the function

$$x^2 + 3x + 2$$

Answer.

$$\text{Let } y = x^2 + 3x + 2$$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

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Q2 Find the second order derivatives of the function

$$x^{20}$$

Answer.

$$\text{Let } y = x^{20}$$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 20 \frac{d}{dx}(x^{19}) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

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Q3 Find the second order derivatives of the function.

$$x \cdot \cos x$$

Answer.

$$\text{Let } y = x \cdot \cos x$$

Then

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x \frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x \sin x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x \sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x \sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x)$$

$$= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x) \right]$$

$$= -\sin x - (\sin x + x \cos x)$$

$$= -2\sin x - x \cos x$$

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Q4 Find the second order derivatives of the function.

$\log x$

Answer.

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

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Q5 Find the second order derivatives of the function.

$x^3 \log x$

Answer.

Let $y = x^3 \log x$

Then

$$\frac{dy}{dx} = \frac{d}{dx} [x^3 \log x] = \log x \cdot \frac{d}{dx}(x^3) + x^3 \cdot \frac{d}{dx}(\log x)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2(1 + 3\log x)$$

$$= (1 + 3\log x) \cdot \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(1 + 3\log x)$$

$$= (1 + 3\log x) \cdot 2x + x^2 \cdot \frac{3}{x}$$

$$= 2x + 6x \log x + 3x$$

$$= 5x + 6x \log x$$

$$= x(5 + 6\log x)$$

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Q6 Find the second order derivatives of the function

$y = e^x \sin 5x$

Answer.

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$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(e^x \sin 5x \right) = \sin 5x \frac{d}{dx} \left(e^x \right) + e^x \frac{d}{dx} (\sin 5x) \\
&= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx} (5x) = e^x \sin 5x + e^x \cos 5x \cdot 5 \\
&= e^x (\sin 5x + 5 \cos 5x) \\
\therefore \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left[e^x (\sin 5x + 5 \cos 5x) \right] \\
&= (\sin 5x + 5 \cos 5x) \cdot \frac{d}{dx} \left(e^x \right) + e^x \cdot \frac{d}{dx} (\sin 5x + 5 \cos 5x) \\
&= (\sin 5x + 5 \cos 5x) \cdot \frac{d}{dx} \left(e^x \right) + e^x \cdot \frac{d}{dx} (\sin 5x + 5 \cos 5x) \\
&= (\sin 5x + 5 \cos 5x) e^x + e^x \left[\cos 5x \cdot \frac{d}{dx} (5x) + 5(-\sin 5x) \cdot \frac{d}{dx} (5x) \right] \\
&= e^x (\sin 5x + 5 \cos 5x) + e^x (5 \cos 5x - 25 \sin 5x) \\
&= e^x (10 \cos 5x - 24 \sin 5x) = 2e^x (5 \cos 5x - 12 \sin 5x)
\end{aligned}$$

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Q7 Find the second order derivatives of the function.

$$e^{6x} \cos 3x$$

Answer.

$$\text{Let } y = e^{6x} \cos 3x$$

Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(e^{6x} \cdot \cos 3x \right) = \cos 3x \cdot \frac{d}{dx} \left(e^{6x} \right) + e^{6x} \cdot \frac{d}{dx} (\cos 3x) \\
&= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx} (6x) + e^{6x} \cdot (-\sin 3x) \cdot \frac{d}{dx} (3x) \\
&= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \\
\therefore \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(6e^{6x} \cos 3x - 3e^{6x} \sin 3x \right) = 6 \cdot \frac{d}{dx} \left(e^{6x} \cos 3x \right) - 3 \cdot \frac{d}{dx} \left(e^{6x} \sin 3x \right) \\
&= 6 \cdot \left[6e^{6x} \cos 3x - 3e^{6x} \sin 3x \right] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx} \left(e^{6x} \right) + e^{6x} \cdot \frac{d}{dx} (\sin 3x) \right] \\
&= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x - 9e^{6x} \cos 3x \\
&= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x \\
&= 9e^{6x} (3 \cos 3x - 4 \sin 3x)
\end{aligned}$$

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Q8 Find the second order derivatives of the function.

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Answer.

$$\text{Let } y = \tan^{-1}x$$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{1}{1+x^2} \right) = \frac{d}{dx} (1+x^2)^{-1} = (-1) \cdot (1+x^2)^{-2} \cdot \frac{d}{dx} (1+x^2) \\ &= \frac{-1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2}\end{aligned}$$

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Q9 Find the second order derivatives of the function.

$\log(\log x)$

Answer.

$$\text{Let } y = \log(\log x)$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[\log(\log x)] = \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) = \frac{1}{x \log x} = (x \log x)^{-1} \\ &= \frac{-1}{(x \log x)^2} \left[\log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x) \right] \\ &= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot 1 + x \cdot \frac{1}{x} \right] = \frac{-(1 + \log x)}{(x \log x)^2}\end{aligned}$$

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Q10

Find the second order derivatives of the function.

$\sin(\log x)$

$$\text{Answer. } \frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)] = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{\cos(\log x)}{x} \right]$$

$$\begin{aligned}
&= \frac{x \cdot \frac{d}{dx}[\cos(\log x)] - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2} \\
&= \frac{x \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x) \right] - \cos(\log x) \cdot 1}{x^2} \\
&= \frac{-x \sin(\log x) \cdot \frac{1}{x} - \cos(\log x)}{x^2} \\
&= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}
\end{aligned}$$

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Q11 If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer.

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x) \\
&= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x) \\
\therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}[-(5\sin x + 3\cos x)]
\end{aligned}$$

$$\begin{aligned}
&= - \left[5 \cdot \frac{d}{dx}(\sin x) + 3 \cdot \frac{d}{dx}(\cos x) \right] \\
&= - [5\cos x + 3(-\sin x)] \\
&= - [5\cos x - 3\sin x]
\end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence proved

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Q12 $y = \cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y

Answer.

It is given that, $y = \cos^{-1}x$

Then,

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$$\frac{dy}{dx} = \frac{d}{dx} (\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}} = - (1-x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[- (1-x^2)^{-\frac{1}{2}} \right]$$

$$= - \left(-\frac{1}{2} \right) \cdot (1-x^2)^{-\frac{3}{2}} \cdot \frac{d}{dx} (1-x^2)$$

$$= \frac{1}{2\sqrt{(1-x^2)^3}} \times (-2x)$$

$$\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}}$$

$$y = \cos^{-1}x \Rightarrow x = \cos y$$

Putting $x = \cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(\sin^2 y)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cot y \cdot \operatorname{cosec}^2 y$$

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Q13 $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

Answer.

It is given that, $y = 3\cos(\log x) + 4\sin(\log x)$

Then,

$$\begin{aligned}
y_1 &= 3 \cdot \frac{d}{dx}[\cos(\log x)] + 4 \cdot \frac{d}{dx}[\sin(\log x)] \\
&= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x) \right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx}(\log x) \right] \\
\therefore y_1 &= \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x} \\
\therefore y_2 &= \frac{d}{dx} \left(\frac{4\cos(\log x) - 3\sin(\log x)}{x} \right) \\
&= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2} \\
&= \frac{x[4\{\cos(\log x)\}' - 3\sin(\log x)'] - \{4\cos(\log x) - 3\sin(\log x)\}}{x^2} \\
&= \frac{x[-4\sin(\log x) \cdot (\log x)' - 3\cos(\log x)(\log x)'] - 4\cos(\log x) + 3\sin(\log x)}{x^2} \\
&= \frac{x[-4\sin(\log x) \cdot (\log x)' - 3\cos(\log x)(\log x)'] - 4\cos(\log x) + 3\sin(\log x)}{x^2} \\
&= \frac{x \left[-4\sin(\log x) \cdot \frac{1}{x} - 3\cos(\log x) \frac{1}{x^2} \right] - 4\cos(\log x) + 3\sin(\log x)}{x^2} \\
&= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2} \\
&= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}
\end{aligned}$$

$$\begin{aligned}
&x^2 y_2 + x y_1 + y \\
&= x^2 \left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2} \right) + x \left(\frac{4\cos(\log x) - 3\sin(\log x)}{x} \right) + 3\cos(\log x) + 4\sin(\log x) \\
&= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)
\end{aligned}$$

Hence, proved.

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Q14 If $y = Ae^{mx} + Be^m$ Show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer It is given that $y = Ae^{mx} + Be^\alpha$

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$$\begin{aligned}\frac{dy}{dx} &= A \cdot \frac{d}{dx}(e^{mx}) + B \cdot \frac{d}{dx}(e^{nx}) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}(Ame^{mx} + Bne^{nx}) = Am \cdot \frac{d}{dx}(e^{mx}) + Bn \cdot \frac{d}{dx}(e^{nx}) \\ &= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx} \\ &= Am^2e^{mx} + Bn^2e^{nx} - (m+n) \cdot (Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx}) \\ &= 0 \text{ Hence proved}\end{aligned}$$

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Q15 If $y = 500e^{7x} + 600e^{-7x}$ Show that $\frac{d^2y}{dx^2} = 49y$

Answer.

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\begin{aligned}\frac{dy}{dx} &= 500 \cdot \frac{d}{dx}(e^{7x}) + 600 \cdot \frac{d}{dx}(e^{-7x}) \\ &= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x) \\ &= 3500e^{7x} - 4200e^{-7x} \\ &= 3500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) - 420 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x) \\ &= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x} \\ &= 49 \times 500e^{7x} + 49 \times 600 \cdot e^{-7x} \\ &= 49(500e^{7x} + 600e^{-7x}) \\ &= 49y\end{aligned}$$

Hence proved.

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Q16 $e^y(x+1) = 1$ Show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Answer.

The given relationship is $e^y(x+1) = 1$

$$e^y(x+1) = 1$$

$$\Rightarrow e^y = \frac{1}{x+1}$$

$$e^y(x+1) = 1$$

$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y = \log \frac{1}{(x+1)}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx} = (x+1) \frac{d}{dx} \left(\frac{1}{x+1} \right) = (x+1) \cdot \frac{-1}{(x+1)^2} = \frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2} = - \frac{d}{dx} \left(\frac{1}{x+1} \right) = - \left(\frac{-1}{(x+1)^2} \right) = \frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{-1}{x+1} \right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx} \right)^2$$

Hence proved

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Q17 if $y = \left(\tan^{-1} x \right)^2$, show that $\left(x^2 + 1 \right)^2 y_2 + 2x \left(x^2 + 1 \right) y_1 = 2$

Answer.

The given relationship is $y = \left(\tan^{-1} x \right)^2$

Then,

$$y_1 = 2 \tan^{-1} x \frac{d}{dx} \left(\tan^{-1} x \right)$$

$$\Rightarrow y_1 = 2 \tan^{-1} x \cdot \frac{1}{1+x^2}$$

$$\Rightarrow \left(1 + x^2 \right) y_1 = 2 \tan^{-1} x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1 + x^2 \right) y_2 + 2x y_1 = 2 \left(\frac{1}{1+x^2} \right)$$

$$\Rightarrow \left(1 + x^2 \right)^2 y_2 + 2x \left(1 + x^2 \right) y_1 = 2$$

Hence proved.

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Q1 Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

Answer. The given function $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in $[-4, 2]$ and is differentiable in $(-4, 2)$.

$$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$$

$$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$$

$$\therefore f(-4) = f(2) = 0$$

$\Rightarrow$ The value of $f(x)$ at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$

$$\Rightarrow f'(x) = 2x + 2$$

$$\therefore f'(c) = 0$$

$$\Rightarrow 2c + 2 = 0$$

$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

Q2 Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

$$f(x) = [x] \text{ for } x \in [5, 9]$$

$$f(x) = [x] \text{ for } x \in [-2, 2]$$

$$(iii) f(x) = x^2 - 1 \text{ for } x \in [1, 2]$$

Answer.

By Rolle's Theorem, for a function $f: [a, b] \rightarrow \mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

then, there exists some $c \in (a, b)$ such that $f'(c) = 0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

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The left hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow \infty} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0} \frac{n-1-n}{h} = \lim_{h \rightarrow \infty} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow \infty} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0} \frac{n-n}{h} = \lim_{h \rightarrow \infty} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$
 $\therefore f$ is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

The left hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0} \frac{n-1-n}{h} = \lim_{h \rightarrow \infty} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow \infty} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0} \frac{n-n}{h} = \lim_{h \rightarrow \infty} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolles Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f , being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

$$f(1) = (1)^2 - 1 = 0$$

$$f(2) = (2)^2 - 1 = 3$$

$$\text{af } f(1) \neq f(2)$$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

$$f(x) = x^2 - 1 \text{ for } x \in [1, 2]$$

If $f: [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$

Answer.

It is given that $f: [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function.

since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\Rightarrow f'(c) \neq 0$$

$$\Rightarrow 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence Proved .

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Q4

Verify Mean value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer.

The given function is $f(x) = x^2 - 4x - 3$

f being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$

whose derivative is $2x - 4$.

$$f(1) = 1^2 - 4 \times 1 - 3 = -6, f(4) = 4^2 - 4 \times 4 - 3 = -3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$$\Rightarrow 2c - 4 = 1$$

$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)$$

Hence, Mean Value Theorem is verified for the given function.

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G5

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Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$

Answer.

The given function f is $f(x) = x^3 - 5x^2 - 3x$

f , being a polynomial function, is continuous in $[1, 3]$ and is differentiable in $(1, 3)$

whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7, f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{-27-(-7)}{3-1} = -10$$

$$f'(c) = -10$$

$$\Rightarrow 3c^2 - 10c - 3 = 10$$

$$\Rightarrow 3c^2 - 10c + 7 = 0$$

$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$

$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$

$$\Rightarrow (c-1)(3c-7) = 0$$

$$\Rightarrow c = 1, \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1, 3)$ is the only point for which $f'(c) = 0$

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Q6 Examine the applicability of Mean Value Theorem for all three functions given in the above Q2.

Answer.

Mean Value Theorem states that for a function $f: [a, b] \rightarrow \mathbf{R}$, if

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b)

then, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$.

The left hand limit of f at $x = n$ is

$$\lim_{h \rightarrow 0} \frac{f(n+h)-f(n)}{h} = \lim_{h \rightarrow 0} \frac{[n+h]-[n]}{h} = \lim_{k \rightarrow 0} \frac{n-1-n}{h} = \lim_{h \rightarrow \infty} \frac{-1}{h} =$$

The right hand limit of f at $x = n$ is,

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since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

if is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0} \frac{n-1-n}{h} = \lim_{h \rightarrow 0} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \rightarrow 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0} \frac{n-n}{h} = \lim_{h \rightarrow 0} 0 = 0$$

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem

$f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f , being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$

It can be proved as follows.

$$f(1) = 1^2 - 1 = 0, f(2) = 2^2 - 1 = 3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3$$

$$\therefore f'(x) = 2x$$

$$\therefore f'(c) = 3$$

$$\Rightarrow c = \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]$$

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Q1 Differentiate w.r.t. x the function in $(3x^2 - 9x + 5)^9$

Answer.

$$\text{Let } y = (3x^2 - 9x + 5)^9$$

Using chain rule, we obtain

$$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 (6x - 9)$$

$$\begin{aligned}
&= 9(3x^2 - 9x + 5)^8 \cdot \frac{d}{dx}(3x^2 - 9x + 5) \\
&= 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) \\
&= 9(3x^2 - 9x + 5)^8 \cdot 3(2x - 3) \\
&= 27(3x^2 - 9x + 5)^8(2x - 3)
\end{aligned}$$

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Q2 Differentiate w.r.t. x the function in $\sin^3 x + \cos^6 x$

Answer.

$$\text{Let } y = \sin^3 x + \cos^6 x$$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\cos^6 x) \\
&= 3\sin^2 x \cdot \frac{d}{dx}(\sin x) + 6\cos^5 x \cdot \frac{d}{dx}(\cos x) \\
&= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x) \\
&= 3\sin x \cos x (\sin x - 2\cos^4 x)
\end{aligned}$$

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Q3 Differentiate w.r.t. x the function in $(5x)^{3\cos 2x}$

Answer.

$$\text{Let } y = (5x)^{\sin 2x}$$

Taking logarithm on both the sides, we obtain

$$\log y = 3\cos 2x \log 5x$$

Differentiating both sides with respect to x, we obtain

$$\begin{aligned}
\frac{1}{y} \frac{dy}{dx} &= 3 \left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x) \right] \\
\Rightarrow \frac{dy}{dx} &= 3y \left[\log 5x (-\sin 2x) \cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x} \cdot \frac{d}{dx}(5x) \right] \\
\Rightarrow \frac{dy}{dx} &= 3y \left[-2\sin 2x \log 5x + \frac{\cos 2x}{x} \right] \\
\Rightarrow \frac{dy}{dx} &= 3y \left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x \right] \\
\therefore \frac{dy}{dx} &= (5x)^{3\cos 2x} \left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x \right]
\end{aligned}$$

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Q4 Differentiate w.r.t. x the function in $\sin^{-1}(x\sqrt{x})$, $0 \leq x \leq 1$

Answer.

$$\text{Let } y = \sin^{-1}(x\sqrt{x})$$

Using chain rule, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \sin^{-1}(x\sqrt{x}) \\
&= \frac{1}{\sqrt{1 - (x\sqrt{x})^2}} \times \frac{d}{dx}(x\sqrt{x}) \\
&= \frac{1}{\sqrt{1 - x^3}} \cdot \frac{d}{dx} \left(x^{\frac{3}{2}} \right) \\
&= \frac{1}{\sqrt{1 - x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}} \\
&= \frac{3\sqrt{x}}{\sqrt{1 - x^3}} \\
&= \frac{3}{2} \sqrt{\frac{x}{1 - x^3}}
\end{aligned}$$

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Q5 Differentiate w.r.t. x the function in $\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$

Answer.

$$\cos^{-1} \frac{x}{2}$$

By quotient rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{2x+7} \frac{d}{dx} \left(\cos^{-1} \frac{x}{2} \right) - \left(\cos^{-1} \frac{x}{2} \right) \frac{d}{dx} (\sqrt{2x+7})}{(\sqrt{2x+7})^2} \\ &= \frac{\sqrt{2x+7} \left[\frac{-1}{\sqrt{1 - \left(\frac{x}{2} \right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{2} \right) \right] - \left(\cos^{-1} \frac{x}{2} \right) \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx} (2x+7)}{dx} (2x+7) \\ &= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2} \times (2x+7)} - \frac{\cos^{-1} \frac{x}{2}}{(\sqrt{2x+7})(2x+7)} \\ &= - \left[\frac{1}{\sqrt{4-x^2} \sqrt{2x+7}} + \frac{2}{(2x+7)^2} \right] \end{aligned}$$

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Q6 Differentiate w.r.t. x the function in $\cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{1}{2}$

Answer.

$$\begin{aligned} \text{Let } y &= \cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right] \\ \text{Then, } \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} &= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x} - \sqrt{1-\sin x} + \sqrt{1-\sin x})^2} \\ &= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x) - (1-\sin x)} \end{aligned}$$

$$\begin{aligned}
& \frac{2 + 2\sqrt{1 - \sin^2 x}}{2\sin x} \\
&= \frac{1 + \cos x}{\sin x} \\
&= \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \\
&= \cot \frac{x}{2} \cos \frac{x}{2} \\
&= \cot \frac{x}{2}
\end{aligned}$$

Therefore, equation (1) becomes

$$\begin{aligned}
y &= \cot^{-1} \left(\cot \frac{x}{2} \right) \\
\Rightarrow y &= \frac{x}{2} \\
\therefore \frac{dy}{dx} &= \frac{1}{2} \frac{d}{dx}(x) \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{2}
\end{aligned}$$

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Q7 Differentiate w.r.t. x the function in $(\log x)^{\log x}$, $x > 1$

Answer. $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\begin{aligned}
\frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} [\log x \cdot \log(\log x)] \\
\Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log(\log x) \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}[\log(\log x)] \\
\Rightarrow \frac{dy}{dx} &= y \left[\log(\log x) \cdot \frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \right] \\
\Rightarrow \frac{dy}{dx} &= y \left[\frac{1}{x} \log(\log x) + \frac{1}{x} \right]
\end{aligned}$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x} \left[\frac{1}{x} + \frac{\log(\log x)}{x} \right]$$

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Q8 Differentiate w.r.t. x the function in $\cos(acosx + bsinx)$, for some constant a and b

Answer.

$$\text{Let } y = \cos(acosx + bsinx)$$

By using chain rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \cos(acosx + bsinx) \\ \Rightarrow \frac{dy}{dx} &= -\sin(acosx + bsinx) \cdot \frac{d}{dx}(acosx + bsinx) \\ &= -\sin(acosx + bsinx) \cdot [a(-\sin x) + b\cos x] \\ &= (asinx - bcosx) \cdot \sin(acosx + bsinx) \end{aligned}$$

Q9 Differentiate w.r.t. x the function in $(\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer.

$$\text{Let } y = (\sin x - \cos x)^{(\sin x - \cos x)}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log \left[(\sin x - \cos x)^{(\sin x - \cos x)} \right]$$

$$\Rightarrow \log y = (\sin x - \cos x) \cdot \log(\sin x - \cos x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} [(\sin x - \cos x) \log(\sin x - \cos x)] \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log(\sin x - \cos x) \cdot \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \cdot \frac{d}{dx} \log(\sin x - \cos x) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log(\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx}(\sin x - \cos x) \\ \Rightarrow \frac{dy}{dx} &= (\sin x - \cos x)^{(\sin x - \cos x)} [(\cos x + \sin x) \cdot \log(\sin x - \cos x) + (\cos x + \sin x)] \\ \therefore \frac{dy}{dx} &= (\sin x - \cos x)^{(\ln x - \cos x)} (\cos x + \sin x) [1 + \log(\sin x - \cos x)] \end{aligned}$$

Q10 $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

$$\text{Answer. } y = x^x + x'' + a^x + a^a$$

Also, let $x^x = u$, $x^a = v$, $a^x = w$, and $a^a = s$

$$\therefore y = u + v + w + s$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx}$$

$$u = x^x$$

$$\Rightarrow \log u = \log x^x$$

$$\Rightarrow \log u = x \log x$$

$$\frac{1}{u} \frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\log x \cdot 1 + x \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^x [\log x + 1] = x^x (1 + \log x)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1}$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a$$

$$s = a^a$$

Since a is constant, a^a is also a constant.

$$\frac{ds}{dx} = 0$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a + 0$$

$$= x^x(1 + \log x) + ax^{a-1} + a^x \log a$$

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Q11 Differentiate w.r.t. x the function in

$$x^{x^2-3} + (x-3)^{x^2}, \text{ for } x > 3$$

Answer

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$$\text{Let } y = x^{x^2-3} + (x-3)^{x^2}$$

$$\text{Also, let } u = x^{x^2-3} \text{ and } v = (x-3)^{x^2}$$

$$\therefore y = u + v$$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = x^{x^2-3}$$

$$\therefore \log u = \log(x^{x^2-3})$$

$$\log u = (x^2 - 3) \log x$$

$$u = x^{x^2-3}$$

$$\therefore \log u = \log(x^{x^2-3})$$

$$\log u = (x^2 - 3) \log x$$

Differentiating with respect to x, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{d}{dx}(x^2 - 3) + (x^2 - 3) \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \log x \cdot 2x + (x^2 - 3) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2} \left[\frac{x^2-3}{x} + 2x \log x \right]$$

Also,

$$v = (x-3)^{x^2}$$

$$\therefore \log v = \log(x-3)^{x^2}$$

$$\Rightarrow \log v = x^2 \log(x-3)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log(x-3) \cdot \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \cdot \frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v \left[2x \log(x-3) + \frac{x^2}{x-3} \cdot 1 \right]$$

$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log(x-3) \right]$$

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Substituting the expressions of dx and $\frac{dv}{dx}$ equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3} \left[\frac{x^2-3}{x} + 2x \log x \right] + (x-3)^x \left[\frac{x^2}{x-3} + 2x \log(x-3) \right]$$

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Q12 $\frac{dy}{dx}$, if $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Answer.

It is given that, $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$

$$\therefore \frac{dx}{dt} = \frac{d}{dt}[10(t - \sin t)] = 10 \cdot \frac{d}{dt}(t - \sin t) = 10(1 - \cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[12(1 - \cos t)] = 12 \cdot \frac{d}{dt}(1 - \cos t) = 12 \cdot [0 - (-\sin t)] = 12 \sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{12 \cdot 2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}}{10 \cdot 2 \sin^2 \frac{t}{2}} = \frac{6}{5} \cot \frac{t}{2}$$

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Q13 Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $-1 \leq x \leq 1$

Answer.

It is given that, $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left[\sin^{-1}x + \sin^{-1}\sqrt{1-x^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\sin^{-1}x \right) + \frac{d}{dx} \left(\sin^{-1}\sqrt{1-x^2} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} (1-x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}} (-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = 0$$

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Q14

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Answer.

It is given that,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

Squaring both sides, we obtain

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + xy^2$$

$$\Rightarrow x^2 - y^2 = xy^2 - x^2y$$

$$\Rightarrow x^2 - y^2 = xy(y - x)$$

$$\Rightarrow (x + y)(x - y) = xy(y - x)$$

$$\therefore x + y = -xy$$

$$\Rightarrow (1 + x)y = -x$$

$$\Rightarrow y = \frac{-x}{(1+x)}$$

Differentiating both sides with respect to x , we obtain

$$y = \frac{-x}{(1+x)}$$

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x) - x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence, proved.

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Q15 if $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$ prove that

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ is a constant independent of } a \text{ and } b.$$

Answer.

It is given that, $(x - a)^2 + (y - b)^2 = c^2$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
\frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] &= \frac{d}{dx}(c^2) \\
\Rightarrow 2(x-a) \cdot \frac{d}{dx}(x-a) + 2(y-b) \cdot \frac{d}{dx}(y-b) &= 0 \\
\Rightarrow 2(x-a) \cdot 1 + 2(y-b) \cdot \frac{dy}{dx} &= 0 \\
\Rightarrow \frac{dy}{dx} &= \frac{-(x-a)}{y-b} \\
\therefore \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\frac{-(x-a)}{y-b} \right] \\
&= - \left[\frac{(y-b) \cdot \frac{d}{dx}(x-a) - (x-a) \cdot \frac{d}{dx}(y-b)}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b) - (x-a) \cdot \frac{dy}{dx}}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b) - (x-a) \cdot \left\{ \frac{-(x-a)}{y-b} \right\}}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right] \\
&= \frac{\left[\frac{c^2}{(y-b)^2} \right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}} \\
&= -c, \text{ which is constant and is independent of } a \text{ and } b
\end{aligned}$$

Hence, proved.

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Q16 $\cos y = x \cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

Processing math: 100%

It is given that, $\cos y = x \cos(a + y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x \cos(a + y)]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a + y) \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}[\cos(a + y)]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a + y) + x \cdot [-\sin(a + y)] \frac{dy}{dx}$$

$$\Rightarrow [x \sin(a + y) - \sin y] \frac{dy}{dx} = \cos(a + y)$$

$$\text{since } \cos y = x \cos(a + y), x = \frac{\cos y}{\cos(a + y)}$$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a + y)} \cdot \sin(a + y) - \sin y \right] \frac{dy}{dx} = \cos(a + y)$$

$$\Rightarrow \sin(a + y - y) \frac{dy}{dx} = \cos^2(a + b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a + b)}{\sin a}$$

Hence, proved.

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Q17 If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$

Answer.

It is given that, $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$

$$\therefore \frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t \sin t)$$

$$= a \left[-\sin t + \sin t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\sin t) \right]$$

$$= a[-\sin t + \sin t + t \cos t] = at \cos t$$

$$= a \left[\cos t - \left\{ \cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t) \right\} \right]$$

$$= a[\cos t - \{\cos t - t \sin t\}] = at \sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{at \sin t}{at \cos t} = \tan t$$

$$\text{Then, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$$

Processing math: 100%

$$= \sec^2 t \cdot \frac{1}{a \cos t} \left[\frac{dx}{dt} = a t \cos t \Rightarrow \frac{dt}{dx} = \frac{1}{a \cos t} \right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

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Q18 $f(x) = |x|^3$, show that $f'(x)$ exists for all real x , and find it.

Answer. It is known that

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f'(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f'(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

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Q19 Using mathematical induction prove that

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ for all positive integer } n.$$

Answer.

$$P(n): \frac{d}{dx}(x^n) = nx^{n-1} \text{ for all positive integers } n$$

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

$\therefore P(n)$ is true for $n = 1$

Let $P(k)$ is true for some positive integer k .

Let $P(k)$ is true for some positive integer k

$$\text{That is, } P(k): \frac{d}{dx}(x^k) = kx^{k-1}$$

It has to be proved that $P(k+1)$ is also true.

$$\text{Consider } \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$$

$$\begin{aligned}
\text{Consider } \frac{d}{dx}(x^{i+1}) &= \frac{d}{dx}(x \cdot x^k) \\
&= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \\
&= x^k \cdot 1 + x \cdot k \cdot x^{k-1} \\
&= x^k + kx^k \\
&= (k+1) \cdot x^k \\
&= (k+1) \cdot x^{(k+1)-1}
\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true. Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer n . Hence proved

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Q20 Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

Answer.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

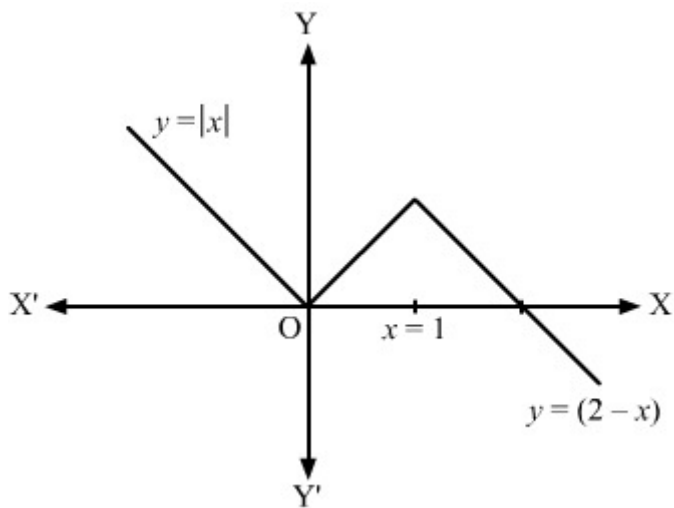
Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
\frac{d}{dx}[\sin(A+B)] &= \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B) \\
\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) &= \cos B = \cos B \frac{d}{dx}(\sin A) + \sin A \cdot \frac{d}{dx}(\cos B) \\
&\quad + \sin B \cdot \frac{d}{dx}(\cos A) + \cos A \cdot \frac{d}{dx}(\sin B) \\
\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) &= \cos B \cdot \cos A \frac{dA}{dx} + \sin A (-\sin B) \frac{dB}{dx} \\
&\quad + \sin B (-\sin A) \frac{dA}{dx} + \cos A \cos B \frac{dB}{dx} \\
\Rightarrow \cos(A+B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx} \right] &= (\cos A \cos B - \sin A \sin B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx} \right] \\
\therefore \cos(A+B) &= \cos A \cos B - \sin A \sin B
\end{aligned}$$

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Q21 Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Answer.



$$y=x$$

$$-\infty < x \leq 1$$

$$2-x$$

$$1 \leq x < \infty$$

It can be seen from the above graph that, the given function is continuous everywhere but not differentiable at exactly two points which are 0 and 1.

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Q22 if $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer.

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y = (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x)$$

$$\text{Then, } \frac{dy}{dx} = \frac{d}{dx}[(mc - nb)f(x)] - \frac{d}{dx}[(lc - na)g(x)] + \frac{d}{dx}[(lb - ma)h(x)]$$

$$= (mc - nb)f'(x) - (lc - na)g'(x) + (lb - ma)h'(x)$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Processing math: 100%

Q23 if $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$ show that

$$\left(1 - x^2\right) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

Answer. It is given that, $y = e^{a \cos^{-1} x}$

Taking logarithm on both the sides, we obtain

$$\log y = a \cos^{-1} x \log e$$

$$\log y = a \cos^{-1} x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$\Rightarrow (1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

$$(1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

Again differentiating both sides with respect to x , we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}(1-x^2) + (1-x^2) \times \frac{d}{dx} \left[\left(\frac{dy}{dx}\right)^2 \right] = a^2 \frac{d}{dx}(y^2)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow -x \frac{dy}{dx} + (1-x^2) \frac{d^2 y}{dx^2} = a^2 \cdot y$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

Hence, proved.

