

Matrices

3.1 Overview

3.1.1 A matrix is an ordered rectangular array of numbers (or functions). For example,

$$A = \begin{bmatrix} x & 4 & 3 \\ 4 & 3 & x \\ 3 & x & 4 \end{bmatrix}$$

The numbers (or functions) are called the elements or the entries of the matrix.

The horizontal lines of elements are said to constitute rows of the matrix and the vertical lines of elements are said to constitute columns of the matrix.

3.1.2 Order of a Matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix).

In the above example, we have A as a matrix of order 3×3 i.e., 3×3 matrix.

In general, an $m \times n$ matrix has the following rectangular array :

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad 1 \leq i \leq m, 1 \leq j \leq n \quad i, j \in \mathbf{N}.$$

The element, a_{ij} is an element lying in the i^{th} row and j^{th} column and is known as the $(i, j)^{\text{th}}$ element of A . The number of elements in an $m \times n$ matrix will be equal to mn .

3.1.3 Types of Matrices

- (i) A matrix is said to be a *row matrix* if it has only one row.

- (ii) A matrix is said to be a *column matrix* if it has only one column.
- (iii) A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix. Thus, an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order ' n '.
- (iv) A square matrix $B = [b_{ij}]_{n \times n}$ is said to be a *diagonal matrix* if its all non diagonal elements are zero, that is a matrix $B = [b_{ij}]_{n \times n}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.
- (v) A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is, a square matrix $B = [b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0, \text{ when } i \neq j$$

$$b_{ij} = k, \text{ when } i = j, \text{ for some constant } k.$$
- (vi) A square matrix in which elements in the diagonal are all 1 and rest are all zeroes is called an identity matrix.
In other words, the square matrix $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.
- (vii) A matrix is said to be *zero matrix* or *null matrix* if all its elements are zeroes. We denote zero matrix by O .
- (ix) Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if
 - (a) they are of the same order, and
 - (b) each element of A is equal to the corresponding element of B, that is, $a_{ij} = b_{ij}$ for all i and j .

3.1.4 Addition of Matrices

Two matrices can be added if they are of the same order.

3.1.5 Multiplication of Matrix by a Scalar

If $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by a scalar k , i.e. $kA = [ka_{ij}]_{m \times n}$

3.1.6 Negative of a Matrix

The negative of a matrix A is denoted by $-A$. We define $-A = (-1)A$.

3.1.7 Multiplication of Matrices

The multiplication of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{\text{th}}$ element c_{ik} of the matrix C , we take the i^{th} row of A and k^{th} column of B , multiply them elementwise and take the sum of all these products i.e.,

$$c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \dots + a_{in} b_{nk}$$

The matrix $C = [c_{ik}]_{m \times p}$ is the product of A and B .

Notes:

1. If AB is defined, then BA need not be defined.
2. If A, B are, respectively $m \times n, k \times l$ matrices, then both AB and BA are defined if and only if $n = k$ and $l = m$.
3. If AB and BA are both defined, it is not necessary that $AB = BA$.
4. If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.
5. For three matrices A, B and C of the same order, if $A = B$, then $AC = BC$, but converse is not true.
6. $A \cdot A = A^2, A \cdot A \cdot A = A^3$, so on

3.1.8 Transpose of a Matrix

1. If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A .

Transpose of the matrix A is denoted by A' or (A^T) . In other words, if $A = [a_{ij}]_{m \times n}$, then $A^T = [a_{ji}]_{n \times m}$.

2. Properties of transpose of the matrices

For any matrices A and B of suitable orders, we have

- (i) $(A^T)^T = A$,
- (ii) $(kA)^T = kA^T$ (where k is any constant)
- (iii) $(A + B)^T = A^T + B^T$
- (iv) $(AB)^T = B^T A^T$

3.1.9 Symmetric Matrix and Skew Symmetric Matrix

- (i) A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$, that is, $a_{ij} = a_{ji}$ for all possible values of i and j .

- (ii) A square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if $A^T = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j .

Note : Diagonal elements of a skew symmetric matrix are zero.

- (iii) **Theorem 1:** For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.
- (iv) **Theorem 2:** Any square matrix A can be expressed as the sum of a symmetric matrix and a skew symmetric matrix, that is

$$A = \frac{(A + A^T)}{2} + \frac{(A - A^T)}{2}$$

3.1.10 Invertible Matrices

- (i) If A is a square matrix of order $m \times m$, and if there exists another square matrix B of the same order $m \times m$, such that $AB = BA = I_m$, then, A is said to be invertible matrix and B is called the inverse matrix of A and it is denoted by A^{-1} .

Note :

1. A rectangular matrix does not possess its inverse, since for the products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
 2. If B is the inverse of A , then A is also the inverse of B .
- (ii) **Theorem 3** (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.
- (iii) **Theorem 4 :** If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

3.1.11 Inverse of a Matrix using Elementary Row or Column Operations

To find A^{-1} using elementary row operations, write $A = IA$ and apply a sequence of row operations on $(A = IA)$ till we get, $I = BA$. The matrix B will be the inverse of A . Similarly, if we wish to find A^{-1} using column operations, then, write $A = AI$ and apply a sequence of column operations on $A = AI$ till we get, $I = AB$.

Note : In case, after applying one or more elementary row (or column) operations on $A = IA$ (or $A = AI$), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

3.2 Solved Examples

Short Answer (S.A.)

Example 1 Construct a matrix $A = [a_{ij}]_{2 \times 2}$ whose elements a_{ij} are given by $a_{ij} = e^{2ix} \sin jx$.

Solution For $i = 1, j = 1, \quad a_{11} = e^{2x} \sin x$
 For $i = 1, j = 2, \quad a_{12} = e^{2x} \sin 2x$
 For $i = 2, j = 1, \quad a_{21} = e^{4x} \sin x$
 For $i = 2, j = 2, \quad a_{22} = e^{4x} \sin 2x$

Thus $A = \begin{bmatrix} e^{2x} \sin x & e^{2x} \sin 2x \\ e^{4x} \sin x & e^{4x} \sin 2x \end{bmatrix}$

Example 2 If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix}$, then

which of the sums $A + B$, $B + C$, $C + D$ and $B + D$ is defined?

Solution Only $B + D$ is defined since matrices of the same order can only be added.

Example 3 Show that a matrix which is both symmetric and skew symmetric is a zero matrix.

Solution Let $A = [a_{ij}]$ be a matrix which is both symmetric and skew symmetric.

Since A is a skew symmetric matrix, so $A' = -A$.

Thus for all i and j , we have $a_{ij} = -a_{ji}$ (1)

Again, since A is a symmetric matrix, so $A' = A$.

Thus, for all i and j , we have

$$a_{ji} = a_{ij} \quad (2)$$

Therefore, from (1) and (2), we get

$$a_{ij} = -a_{ij} \text{ for all } i \text{ and } j$$

or $2a_{ij} = 0$,

i.e., $a_{ij} = 0$ for all i and j . Hence A is a zero matrix.

Example 4 If $\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = O$, find the value of x .

Solution We have

$$\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = O \Rightarrow \begin{bmatrix} 2x-9 & 4x \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = [0]$$

$$\text{or} \quad \begin{bmatrix} 2x^2 - 9x & +32x \end{bmatrix} = [0] \Rightarrow 2x^2 + 23x = 0$$

$$\text{or} \quad x(2x+23)=0 \quad \Rightarrow \quad x=0, x=\frac{-23}{2}$$

Example 5 If A is 3×3 invertible matrix, then show that for any scalar k (non-zero),

kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

Solution We have

$$(kA) \left(\frac{1}{k}A^{-1} \right) = \left(k, \frac{1}{k} \right) (A, A^{-1}) = I \quad (I) = I$$

$$\text{Hence } (kA) \text{ is inverse of } \left(\frac{1}{k}A^{-1} \right) \quad \text{or} \quad (kA)^{-1} = \frac{1}{k}A^{-1}$$

Long Answer (L.A.)

Example 6 Express the matrix A as the sum of a symmetric and a skew symmetric matrix, where

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}.$$

Solution We have

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}, \quad \text{then } A' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

Hence
$$\frac{A+A'}{2} = \frac{1}{2} \begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & \frac{11}{2} & \frac{-5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 4 \end{bmatrix}$$

and
$$\frac{A-A'}{2} = \frac{1}{2} \begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-3}{2} & \frac{-7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & \frac{-7}{2} & 0 \end{bmatrix}$$

Therefore,

$$\frac{A+A'}{2} + \frac{A-A'}{2} = \begin{bmatrix} 2 & \frac{11}{2} & \frac{-5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-3}{2} & \frac{-7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & \frac{-7}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} = A.$$

Example 7 If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, then show that A satisfies the equation

$$A^3 - 4A^2 - 3A + 11I = O.$$

Solution
$$A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

and $A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

Now $A^3 - 4A^2 - 3A + 11(I)$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4 \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3 \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 8 Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$. Then show that $A^2 - 4A + 7I = O$.

Using this result calculate A^5 also.

Solution We have $A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$,

$$-4A = \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} \text{ and } 7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

$$\text{Therefore, } A^2 - 4A + 7I = \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$\Rightarrow A^2 = 4A - 7I$$

$$\begin{aligned} \text{Thus } A^3 &= A \cdot A^2 = A(4A - 7I) = 4(4A - 7I) - 7A \\ &= 16A - 28I - 7A = 9A - 28I \end{aligned}$$

and so

$$\begin{aligned} A^5 &= A^3 A^2 \\ &= (9A - 28I)(4A - 7I) \\ &= 36A^2 - 63A - 112A + 196I \\ &= 36(4A - 7I) - 175A + 196I \\ &= -31A - 56I \end{aligned}$$

$$= -31 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$

Objective Type Questions

Choose the correct answer from the given four options in Examples 9 to 12.

Example 9 If A and B are square matrices of the same order, then

$(A + B)(A - B)$ is equal to

- (A) $A^2 - B^2$ (B) $A^2 - BA - AB - B^2$
 (C) $A^2 - B^2 + BA - AB$ (D) $A^2 - BA + B^2 + AB$

Solution (C) is correct answer. $(A + B)(A - B) = A(A - B) + B(A - B)$
 $= A^2 - AB + BA - B^2$

Example 10 If $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$, then

- (A) only AB is defined (B) only BA is defined
 (C) AB and BA both are defined (D) AB and BA both are not defined.

Solution (C) is correct answer. Let $A = [a_{ij}]_{2 \times 3}$, $B = [b_{ij}]_{3 \times 2}$. Both AB and BA are defined.

Example 11 The matrix $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ is a

- (A) scalar matrix (B) diagonal matrix
 (C) unit matrix (D) square matrix

Solution (D) is correct answer.

Example 12 If A and B are symmetric matrices of the same order, then $(AB' - BA')$ is a

- (A) Skew symmetric matrix (B) Null matrix
 (C) Symmetric matrix (D) None of these

Solution (A) is correct answer since

$$(AB' - BA')' = (AB')' - (BA')'$$

$$\begin{aligned}
 &= (BA' - AB') \\
 &= - (AB' - BA')
 \end{aligned}$$

Fill in the blanks in each of the Examples 13 to 15:

Example 13 If A and B are two skew symmetric matrices of same order, then AB is symmetric matrix if _____.

Solution $AB = BA$.

Example 14 If A and B are matrices of same order, then $(3A - 2B)'$ is equal to _____.

Solution $3A' - 2B'$.

Example 15 Addition of matrices is defined if order of the matrices is _____

Solution Same.

State whether the statements in each of the Examples 16 to 19 is true or false:

Example 16 If two matrices A and B are of the same order, then $2A + B = B + 2A$.

Solution True

Example 17 Matrix subtraction is associative

Solution False

Example 18 For the non singular matrix A, $(A')^{-1} = (A^{-1})'$.

Solution True

Example 19 $AB = AC \Rightarrow B = C$ for any three matrices of same order.

Solution False

3.3 EXERCISE

Short Answer (S.A.)

1. If a matrix has 28 elements, what are the possible orders it can have? What if it has 13 elements?

2. In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 - y \\ 0 & 5 & \frac{-2}{5} \end{bmatrix}$, write :

- (i) The order of the matrix A
 (ii) The number of elements
 (iii) Write elements a_{23} , a_{31} , a_{12}
3. Construct $a_{2 \times 2}$ matrix where
- (i) $a_{ij} = \frac{(i-2j)^2}{2}$
 (ii) $a_{ij} = |-2i + 3j|$
4. Construct a 3×2 matrix whose elements are given by $a_{ij} = e^{i \cdot x} \sin jx$
5. Find values of a and b if $A = B$, where

$$A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}$$

6. If possible, find the sum of the matrices A and B, where $A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 3 \end{bmatrix}$,

$$\text{and } B = \begin{bmatrix} x & y & z \\ a & b & 6 \end{bmatrix}$$

7. If $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$, find
- (i) $X+Y$ (ii) $2X-3Y$
 (iii) A matrix Z such that $X+Y+Z$ is a zero matrix.
8. Find non-zero values of x satisfying the matrix equation:

$$x \begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2 \begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2 \begin{bmatrix} (x^2+8) & 24 \\ (10) & 6x \end{bmatrix}.$$

9. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that $(A+B)(A-B) \neq A^2 - B^2$.

10. Find the value of x if

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = \mathbf{O}.$$

11. Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $A^2 - 3A - 7I = \mathbf{O}$ and hence find A^{-1} .

12. Find the matrix A satisfying the matrix equation:

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

13. Find A , if $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

14. If $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then verify $(BA)^2 \neq B^2A^2$

15. If possible, find BA and AB , where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

16. Show by an example that for $A \neq \mathbf{O}$, $B \neq \mathbf{O}$, $AB = \mathbf{O}$.

17. Given $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$. Is $(AB)' = B'A'$?

18. Solve for x and y :

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = \mathbf{O}.$$

19. If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}.$$

20. If $A = \begin{bmatrix} 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 3 \end{bmatrix}$, then find a non-zero matrix C such that $AC = BC$.

21. Give an example of matrices A , B and C such that $AB = AC$, where A is non-zero matrix, but $B \neq C$.

22. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, verify :

$$(i) (AB)C = A(BC) \quad (ii) A(B + C) = AB + AC.$$

23. If $P = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ and $Q = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, prove that

$$PQ = \begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix} = QP.$$

24. If : $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = A$, find A .

25. If $A = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, verify that $A(B + C) = (AB + AC)$.

26. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, then verify that $A^2 + A = A(A + I)$, where I is 3×3 unit matrix.

27. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$, then verify that :

- (i) $(A')' = A$
- (ii) $(AB)' = B'A'$
- (iii) $(kA)' = (kA')$.

28. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}$, then verify that :

- (i) $(2A + B)' = 2A' + B'$
- (ii) $(A - B)' = A' - B'$.

29. Show that $A'A$ and AA' are both symmetric matrices for any matrix A .

30. Let A and B be square matrices of the order 3×3 . Is $(AB)^2 = A^2 B^2$? Give reasons.

31. Show that if A and B are square matrices such that $AB = BA$, then

$$(A + B)^2 = A^2 + 2AB + B^2.$$

32. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ and $a = 4$, $b = -2$.

Show that:

- (a) $A + (B + C) = (A + B) + C$
- (b) $A(BC) = (AB)C$

$$(c) \quad (a + b)B = aB + bB$$

$$(d) \quad a(C - A) = aC - aA$$

$$(e) \quad (A^T)^T = A$$

$$(f) \quad (bA)^T = b A^T$$

$$(g) \quad (AB)^T = B^T A^T$$

$$(h) \quad (A - B)C = AC - BC$$

$$(i) \quad (A - B)^T = A^T - B^T$$

$$33. \quad \text{If } A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then show that } A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}.$$

$$34. \quad \text{If } A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } x^2 = -1, \text{ then show that } (A + B)^2 = A^2 + B^2.$$

$$35. \quad \text{Verify that } A^2 = I \text{ when } A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}.$$

36. Prove by Mathematical Induction that $(A^n)' = (A')^n$, where $n \in \mathbf{N}$ for any square matrix A .

37. Find inverse, by elementary row operations (if possible), of the following matrices

$$(i) \quad \begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix} \qquad (ii) \quad \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

$$38. \quad \text{If } \begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}, \text{ then find values of } x, y, z \text{ and } w.$$

$$39. \quad \text{If } A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}, \text{ find a matrix } C \text{ such that } 3A + 5B + 2C \text{ is a null matrix.}$$

40. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, then find $A^2 - 5A - 14I$. Hence, obtain A^3 .

41. Find the values of a, b, c and d , if

$$3 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix}.$$

42. Find the matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}.$$

43. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, find $A^2 + 2A + 7I$.

44. If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, and $A^{-1} = A'$, find value of α .

45. If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew symmetric matrix, find the values of a, b and c .

46. If $P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then show that

$$P(x) \cdot P(y) = P(x+y) = P(y) \cdot P(x).$$

47. If A is square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.

48. If A, B are square matrices of same order and B is a skew-symmetric matrix, show that $A'BA$ is skew symmetric.

Long Answer (L.A.)

49. If $AB = BA$ for any two square matrices, prove by mathematical induction that $(AB)^n = A^n B^n$.

50. Find x, y, z if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

51. If possible, using elementary row transformations, find the inverse of the following matrices

(i) $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

52. Express the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises 53 to 67.

53. The matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a

- (A) square matrix (B) diagonal matrix
(C) unit matrix (D) none

54. Total number of possible matrices of order 3×3 with each entry 2 or 0 is

- (A) 9 (B) 27 (C) 81 (D) 512

55. If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of $x+y$ is

- (A) $x=3, y=1$ (B) $x=2, y=3$
(C) $x=2, y=4$ (D) $x=3, y=3$

56. If $A = \frac{1}{\pi} \begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$, $B = \frac{1}{\pi} \begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$, then

$A - B$ is equal to

- (A) I (B) O (C) $2I$ (D) $\frac{1}{2}I$

57. If A and B are two matrices of the order $3 \times m$ and $3 \times n$, respectively, and $m = n$, then the order of matrix $(5A - 2B)$ is

- (A) $m \times 3$ (B) 3×3 (C) $m \times n$ (D) $3 \times n$

58. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to

- (A) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
 (C) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

59. If matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = 1$ if $i \neq j$
 $= 0$ if $i = j$

then A^2 is equal to

- (A) I (B) A (C) O (D) None of these

60. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a

- (A) identity matrix (B) symmetric matrix
 (C) skew symmetric matrix (D) none of these

61. The matrix $\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$ is a
- (A) diagonal matrix (B) symmetric matrix
(C) skew symmetric matrix (D) scalar matrix
62. If A is matrix of order $m \times n$ and B is a matrix such that AB' and $B'A$ are both defined, then order of matrix B is
- (A) $m \times m$ (B) $n \times n$
(C) $n \times m$ (D) $m \times n$
63. If A and B are matrices of same order, then $(AB' - BA')$ is a
- (A) skew symmetric matrix (B) null matrix
(C) symmetric matrix (D) unit matrix
64. If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to
- (A) A (B) $I - A$ (C) $I + A$ (D) $3A$
65. For any two matrices A and B, we have
- (A) $AB = BA$ (B) $AB \neq BA$
(C) $AB = O$ (D) None of the above
66. On using elementary column operations $C_2 \rightarrow C_2 - 2C_1$ in the following matrix equation

$$\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \text{ we have :}$$

$$(A) \quad \begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$$

$$(B) \quad \begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -0 & 2 \end{bmatrix}$$

$$(C) \quad \begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

$$(D) \quad \begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$$

67. On using elementary row operation $R_1 \rightarrow R_1 - 3R_2$ in the following matrix equation:

$$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \text{ we have :}$$

$$(A) \quad \begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$(B) \quad \begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$$

$$(C) \quad \begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$(D) \quad \begin{bmatrix} 4 & 2 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Fill in the blanks in each of the Exercises 68–81.

68. _____ matrix is both symmetric and skew symmetric matrix.
69. Sum of two skew symmetric matrices is always _____ matrix.
70. The negative of a matrix is obtained by multiplying it by _____.
71. The product of any matrix by the scalar _____ is the null matrix.
72. A matrix which is not a square matrix is called a _____ matrix.
73. Matrix multiplication is _____ over addition.
74. If A is a symmetric matrix, then A^3 is a _____ matrix.
75. If A is a skew symmetric matrix, then A^2 is a _____.

76. If A and B are square matrices of the same order, then
- $(AB)' = \underline{\hspace{2cm}}$.
 - $(kA)' = \underline{\hspace{2cm}}$. (k is any scalar)
 - $[k(A - B)]' = \underline{\hspace{2cm}}$.
77. If A is skew symmetric, then kA is a $\underline{\hspace{2cm}}$. (k is any scalar)
78. If A and B are symmetric matrices, then
- $AB - BA$ is a $\underline{\hspace{2cm}}$.
 - $BA - 2AB$ is a $\underline{\hspace{2cm}}$.
79. If A is symmetric matrix, then $B'AB$ is $\underline{\hspace{2cm}}$.
80. If A and B are symmetric matrices of same order, then AB is symmetric if and only if $\underline{\hspace{2cm}}$.
81. In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeros in one or more, then A^{-1} $\underline{\hspace{2cm}}$.

State Exercises 82 to 101 which of the following statements are **True** or **False**

- A matrix denotes a number.
- Matrices of any order can be added.
- Two matrices are equal if they have same number of rows and same number of columns.
- Matrices of different order can not be subtracted.
- Matrix addition is associative as well as commutative.
- Matrix multiplication is commutative.
- A square matrix where every element is unity is called an identity matrix.
- If A and B are two square matrices of the same order, then $A + B = B + A$.
- If A and B are two matrices of the same order, then $A - B = B - A$.
- If matrix $AB = O$, then $A = O$ or $B = O$ or both A and B are null matrices.
- Transpose of a column matrix is a column matrix.
- If A and B are two square matrices of the same order, then $AB = BA$.
- If each of the three matrices of the same order are symmetric, then their sum is a symmetric matrix.

95. If A and B are any two matrices of the same order, then $(AB)' = A'B'$.
96. If $(AB)' = B'A'$, where A and B are not square matrices, then number of rows in A is equal to number of columns in B and number of columns in A is equal to number of rows in B.
97. If A, B and C are square matrices of same order, then $AB = AC$ always implies that $B = C$.
98. AA' is always a symmetric matrix for any matrix A.
99. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then AB and BA are defined and equal.
100. If A is skew symmetric matrix, then A^2 is a symmetric matrix.
101. $(AB)^{-1} = A^{-1} \cdot B^{-1}$, where A and B are invertible matrices satisfying commutative property with respect to multiplication.



