15 Limit - Continuity - Differentiation

Important Definitions and Necessary Formulae :

(1) Modulus function : $f : R \to R, f(x) = |x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$ is called modulus function.

Properties : (i) $|x + y| \leq |x| + |y|$ (ii) $|x - y| \geq ||x| - |y||$ (iii) $|x| < a \Leftrightarrow -a < x < a$, $a \in R^+$

(2) Integer part function (Floor function) : $f : R \to Z$, $f(x) = [x]$ = the largest integer that is less than or equal to x. (not exceeding x)

This is integer part function.

$f(3.2) = [3.2] = 3$, $f(1) = [1] = 1$, $f(-1{\cdot}3) = [-1.3] = -2$

(3) Least integer function : $f : R \to Z$, $f(x) = \lceil x \rceil$ = the smallest integer greater than or equal to x.

This is smallest integer function.

$f(5.3) = \lceil 5.3 \rceil = 6$, $f(-1.9) = \lceil -1.9 \rceil = -1$, $f(4) = \lceil 4 \rceil = 4$, $f(-2.3) = \lceil -2.3 \rceil = -2$

Graphs of some real functions :

(1) Modulus function : $f : R \to R, f(x) = |x|$

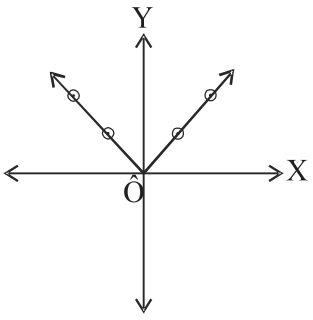


Figure 15.1

(2) Integer part function : $f : R \to Z, f(x) = [x]$

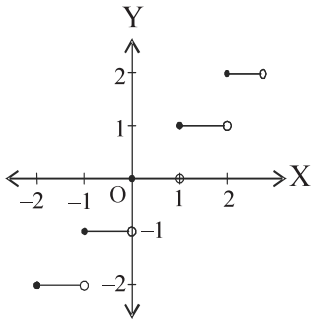


Figure 15.2

(3) Exponential function : $f : R \to R, f(x) = a^x$ where $a > 0$, $a \neq 1$, $a > 1$

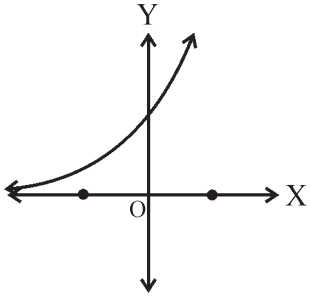


Figure 15.3

(4) Logarithmic function : $f : R^+ \to R, f(x) = \log_a x$ where $a > 0$, $a \neq 1$, $a > 1$

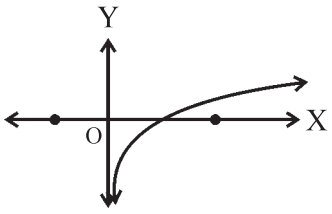


Figure 15.4

Left hand limit : A function f is defined on (c, a). If for a given $\varepsilon > 0$ there corresponds $\delta > 0$ such that $c < a - \delta$ and $\forall x,\ x \in (a - \delta, a) \Rightarrow |f(x) - l| < \varepsilon$, then we say that f has left limit l and it is denoted by $\lim\limits_{x \to a-} f(x) = l$.

Right hand limit : A function f is defined on (a, b). If for a given $\varepsilon > 0$ there corresponds $\delta > 0$ such that $a + \delta < b$ and $\forall x,\ x \in (a,\ a + \delta) \Rightarrow |f(x) - l| < \varepsilon$, then we say that f has right limit l and is denoted by $\lim\limits_{x \to a+} f(x) = l$.

- $\lim\limits_{x \to a} f(x) = l \Leftrightarrow \lim\limits_{x \to a-} f(x) = l = \lim\limits_{x \to a+} f(x)$
- If $\lim\limits_{x \to a} f(x)$ exists, then it is unique.
- If $\lim\limits_{x \to a} f(x) = f(a)$, then the function f is said to be continuous at $x = a$.
- If a function $f(x)$ is continuous at every point $x \in (a,\ b)$, then f is said to be continuous on $(a,\ b)$.

  Moreover, if $\lim\limits_{x \to a+} f(x) = f(a)$ and $\lim\limits_{x \to b\_} f(x) = f(b)$, then f is continuous on $[a,\ b]$.

Working rules of limit :

If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

(1) $\lim\limits_{x \to a} (f(x) \pm g(x)) = l \pm m$

(2) $\lim\limits_{x \to a} k f(x) = kl$

(3) $\lim\limits_{x \to a} f(x)\ g(x) = l \cdot m$

(4) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m},\ m \neq 0$

- $\lim\limits_{x \to a} x^n = a^n, \quad n \in \mathrm{N}$
- If $p(x)$ is a polynomial function, then $\lim\limits_{x \to a} p\ (x) = p(a)$
- A polynomial function is continuous on R.
- A rational function is continuous at every point of its domain.
- $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1},\ n \in \mathrm{N},\ x \in \mathrm{R} - \{a\}$

Rule of limit of composite function :

If $\lim\limits_{x \to a} f\ (x) = b$ and $\lim\limits_{y \to b} g(y) = l$, then $\lim\limits_{x \to a} g\ (f\ (x)) = l$.

- If function g is continuous at b and $\lim\limits_{x \to a} f\ (x) = b$, then

$$\lim_{x \to a} g\ (f(x)) = \mathrm{g}\ (b) = g\left(\lim_{x \to a} f\ (x)\right)$$

If the 'outer' function is continuous, then we can take limit 'inside' :

- If $f(x) < g(x)$ and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

  Sandwich theorem : If functions f, g, h are such that $g \le f \le h$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = l$, then $\lim_{x\to a} f(x) = l$.

- $\lim_{x\to a} f(x)) = 0 \Leftrightarrow \lim_{x\to a} |f(x)| = 0$
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- *sin* function and *cos* function are continuous.
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$
- If f and g are continuous at $x = c$, then $f \pm g$, fg and if $g(c) \neq 0$, then $\frac{f}{g}$ are also continuous $x = c$.

  Limit of a sequence : Let $\{a_n\}$ be a sequence and l be a real number. For a given $\varepsilon > 0$, if there exists $m \in N$ such that $n \ge m$, $n \in N \Rightarrow |a_n - l| < \varepsilon$, then l is the limit of sequence $\{a_n\}$. It is denoted by $\lim_{n\to\infty} a_n = l$.

- $\lim_{n\to\infty} \frac{1}{n} = 0$
- If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$
- Let $S_n = a + ar + ar^2 + ...+ ar^{n-1}$

  $\lim_{n\to\infty} S_n = \frac{a}{1-r}$, $|r| < 1$

- $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$
- $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e \qquad (a > 0)$
- $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

- If $\lim_{x\to a} f(x)$ exists and is positive, then $\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a} g(x)\log f(x)}$
- If $\lim_{x\to a} f(x) = 1$ and $g(x) \to \infty$ then, $\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a} (f(x)-1)\, g(x)}$
- $\lim_{n\to\infty} (\sin^n x + \cos^n x)^{\frac{1}{n}} = \max\{|\sin x|, |\cos x|\}$

Useful infinite series to find the value of limits :

(1) $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots \infty \quad (|x| < 1,\ m \in Q)$

(2) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$

(3) $a^x = 1 + x\,(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots \infty$

(4) $\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty \quad (|x| < 1)$

(5) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \infty$

(6) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \infty$

(7) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \infty$

(8) $\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \ldots \infty \quad (|x| < 1)$

(9) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \infty \quad (|x| < 1)$

(10) $\Sigma n = \frac{n(n+1)}{2}$

(11) $\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

(12) $\Sigma n^3 = \frac{n^2(n+1)^2}{4}$

(13) $\Sigma n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Differentiation : Let $f : (a, b) \to R$ be a function. Let $x \in (a, b)$ be a constant and $t \in (a, b)$ be a variable.

If $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ exists, then we say f is differentiable at x. This limit is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$. For $y = f(x)$, we can also write $\frac{dy}{dx}$ instead of $\frac{d}{dx} f(x)$. It is also denoted by y_1.

The process of obtaining derivative is called differentiation.

- At the point $(x, f(x))$ of a curve $y = f(x)$, $f'(x)$ is the slope of the tangent, if tangent is not vertical.
- If a function $f : (a, b) \to R$, is differentiable at $x \in (a, b)$, then it is continuous there. The converse is not true. $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

L'Hospital rule to obtain limit :

(1) $\left(\frac{0}{0} \text{ form}\right)$ If functions f and g are differentiable and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(2) $\left(\frac{\infty}{\infty} \text{ form}\right)$ If functions f and g are differentiable and as $x \to a, f(x) \to \infty, g(x) \to \infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Working rules of derivatives :

If functions f and g are differentiable and $k \in R$, then

(1) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

(2) $\frac{d}{dx}(k f(x)) = k \frac{d}{dx} f(x)$

(3) $\frac{d}{dx}(f(x)\, g(x)) = g(x) \frac{d}{dx} f(x) + f(x) \frac{d}{dx} g(x)$

(4) If $g(x) \neq 0$, then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

Some standard forms :

(1) $\frac{d}{dx} c = 0$, where c is constant.

(2) $\frac{d}{dx} x^n = nx^{n-1}, x \in R^+, n \in R$

(3) $\frac{d}{dx} a^x = a^x \log a, x \in R, a \in R^+ - \{1\}$

(4) $\frac{d}{dx} e^x = e^x, x \in R$

(5) $\frac{d}{dx} \sin x = \cos x, x \in R$

(6) $\frac{d}{dx} \cos x = -\sin x, x \in R$

(7) $\frac{d}{dx} \tan x = \sec^2 x, x \in R - \{(2k+1) \mid \frac{\pi}{2} \mid k \in Z\}$

(8) $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x, x \in R - \{k\pi \mid k \in Z\}$

(9) $\frac{d}{dx} \sec x = \sec x \tan x, x \in R - \{(2k+1) \frac{\pi}{2} \mid k \in Z\}$

(10) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x, x \in R - \{k\pi \mid k \in Z\}$

(11) $\frac{d}{dx} \log_e x = \frac{1}{x}$

(12) If $f : (a, b) \to R$ is differentiable at $x \in (a, b)$ and $f((a, b)) \subset (c, d)$ and $g : (c, d) \to R$ is differentiable at $f(x)$, then gof is also differentiable at x and $(gof)'(x) = g'(f(x))\, f'(x)$, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. (Chain Rule)

(13) $\frac{d}{dx}\ sin^{-1}\ x = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$

(14) $\frac{d}{dx}\ cos^{-1}\ x = \frac{-1}{\sqrt{1-x^2}}$, $|x| < 1$

(15) $\frac{d}{dx}\ tan^{-1}\ x = \frac{1}{1+x^2}$, $x \in R$

(16) $\frac{d}{dx}\ cot^{-1}\ x = \frac{-1}{1+x^2}$, $x \in R$

(17) $\frac{d}{dx}\ sec^{-1}\ x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

(18) $\frac{d}{dx}\ cosec^{-1}\ x = \frac{-1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

(19) **Differentiation of parametric functions :**

If $x = f(t)$, $y = g(t)$, $t \in [a, b]$ are functions of t,

then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; $f'(t) \neq 0$

(20) **Derivative of one function with respect to another function :** $u = f(x)$ and $v = g(x)$ are differentiable functions of x. Derivative of u with respect to v is $\frac{du}{dv} = \frac{du/dx}{dv/dx}$.

Second derivative :

If $y = f(x)$ is twice differentiable, then $\frac{d}{dx}\ f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. Symbolically we can write second derivative as $f''(x)$ or y_2.

Applications of derivative :

- **Derivative as a rate of change :** If a variable y depends on another variable x by $y = f(x)$, then the rate of change of y with respect to x is $\frac{dy}{dx}$ or $f'(x)$.

Application of derivative to geometry :

(1) For the curve $y = f(x)$ the equation of the tangent at $(x_0, f(x_0))$, $y - y_0 = f'(x_0)\ (x - x_0)$.

The line perpendicular to the tangent passing through the point of contact is called a normal. If the tangent is not parallel or perpendicular to X-axis, then the equation of normal at $(x_0, f(x_0))$ is $y - y_0 = -\frac{1}{f'(x_0)}\ (x - x_0)$.

- If $x \to x_0$ and $f'(x_0) \to \pm\infty$, then for the curve $y = f(x)$ the vertical tangent at $(x_0, f(x_0))$ is $x = x_0$ and normal is $y = y_0$.

(2) Angle between two curves : If two curves $y = f(x)$ and $y = g(x)$ intersect at (x_0, y_0) and f, g are differentiable at (x_0, y_0) then $tan^{-1}\left|\frac{f'(x_0) - g'(x_0)}{1 + f'(x_0)\,g'(x_0)}\right|$ is the measure of the angle between the two curves. If $f'(x_0)\ g'(x_0) = -1$ then two curves intersect at right angles and we say they are orthogonal to each other.

Rolle's theorem :

If (i) f is continuous on $[a, b]$, (ii) f is differentiable on (a, b) and (iii) $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Geometrically, for some $c \in (a, b)$ the tangent at $(c, f(c))$ is either X-axis or parallel to X-axis.

Mean value theorem : If f is continuous on $[a, b]$ and differentiable on (a, b), then there exist $c \in (a, b)$ such that,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Geometrically for some $c \in (a, b)$, the tangent at the point $(c, f(c))$ is parallel to the line joining A $(a, f(a))$ and B $(b, f(b))$.

Let $b = a + h$. Then $f(a + h) - f(a) = h f'(c)$

$\therefore$ We can take $c = a + \theta h$, $0 < \theta < 1$

$\therefore f(a + h) - f(a) = h f'(a + \theta h)$, $0 < \theta < 1$ is another form of mean value theorem.

Increasing and decreasing functions :

Let a function f be defined on an interval (a, b).

If (i) $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ then f is called an increasing function on (a,b). It is denoted by $f \uparrow$.

(ii) $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ then f is called a strictly increasing function on (a,b).

(iii) $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ then f is called a decreasing function on (a,b). It is denoted by $f \downarrow$.

(iv) $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ then f is called a strictly decreasing function on (a,b).

- $f : (a, b) \to R$ is a differentiable function. If (i) $f'(x) \geq 0$, $\forall\ x \in (a, b)$, then f is an increasing function on (a,b). (ii) $f'(x) \leq 0$, $\forall\ x \in (a, b)$, then f is a decreasing function on (a,b).

Maximum and minimum values :

(1) Let a function f be defined on $[a,b]$ and be continuous. If there exists $c \in [a, b]$ such that $f(x) \leq f(c)$, $\forall\ x \in [a, b]$, then $f(c)$ is called a global maximum value of f on $[a, b]$. If $f(x) \geq f(c)$, then $f(c)$ is called a global minimum value of f on $[a, b]$.

(2) Let a function f be defined on $[a,b]$. If there exist $c \in (a, b)$, $h > 0$ such that $(c - h, c + h) \subset (a, b)$ and for every $x \in (c - h, c + h)$, $f(x) \leq f(c)$, then $f(c)$ is called a local maximum value at c. If $f(x) \geq f(c)$, $\forall\ x \in (c - h, c + h)$ then $f(c)$ is called a local minimum value at c.

- If f is defined on (a, b) and has extreme value (minimum or maximum) at c and f is differentiable at c, then $f'(c) = 0$.

(First derivative test) If f is defined on $[a, b]$ and $c \in (a, b)$, $h > 0$ such that $(c - h, c + h) \subset (a, b)$. If,

(*a*) (i) f is differentiable on $(c - h, c + h)$.

(ii) $c < x < c + h \Rightarrow f'(x) > 0$

(iii) $c - h < x < c \Rightarrow f'(x) < 0$

(iv) $f'(c) = 0$ then f has local minimum value at $x = c$. Similarly, if

(*b*) (i) f is differentiable on $(c - h, c + h)$.

(ii) $c - h < x < c \Rightarrow f'(x) > 0$

(iii) $c < x < c + h \Rightarrow f'(x) < 0$

(iv) $f'(c) = 0$ then f has local maximum value at $x = c$.

(Second derivative test)

Let f be defined on $[a, b]$ and $c \in (a, b)$. Let, $h > 0$ and $(c - h, c + h) \subset (a, b)$ and $f'(c) = 0, f''(c) > 0$, then f has local minimum value at c. $f'(c) = 0$ and $f''(c) < 0$, then f has local maximum value at c.

: Multiple Choice Questions :

(1) $\lim_{n\to\infty}\left\{\tan\frac{x}{2}\sec x+\tan\frac{x}{2^2}\sec\frac{x}{2}+\ldots+\tan\left(\frac{x}{2^n}\right)\sec\left(\frac{x}{2^{n-1}}\right)\right\}$

(A) $\cot x$ (B) $\sec x$ (C) $\tan x$ (D) 0

Sol. : $\tan\frac{x}{2}\sec x = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}\cos x} = \frac{\sin\left(x-\frac{x}{2}\right)}{\cos\frac{x}{2}\cos x} = \tan x - \tan\frac{x}{2}$

$\therefore \lim_{n\to\infty}\left\{\tan\frac{x}{2}\sec x+\tan\frac{x}{2^2}\sec\frac{x}{2}+\ldots+\tan\left(\frac{x}{2^n}\right)\sec\left(\frac{x}{2^{n-1}}\right)\right\}$

$= \lim_{n\to\infty}\left\{\left(\tan x-\tan\frac{x}{2}\right)+\left(\tan\frac{x}{2}-\tan\frac{x}{2^2}\right)+\ldots+\left(\tan\frac{x}{2^{n-1}}-\tan\frac{x}{2^n}\right)\right\}$

$= \lim_{n\to\infty}\left(\tan x-\tan\frac{x}{2^n}\right) = \tan x \qquad \left(\lim_{n\to\infty}\frac{1}{2^n}=0\right)$ **Ans. :** (C)

(2) If $f(x) = \begin{cases} \frac{\sin(a+1)x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}}, & x>0 \end{cases}$ is continuous at $x = 0$, then ..... .

(A) $a = \frac{-3}{2}, b = 0, c = \frac{1}{2}$ (B) $a = -\frac{3}{2}, b = 1, c = \frac{1}{2}$

(C) $a = \frac{-3}{2}, b \in R, c = \frac{-1}{2}$ (D) $a = \frac{-3}{2}, b \in R, c = \frac{1}{2}$

Sol. : $\lim_{x\to0-} f(x) = \lim_{x\to0-}\frac{\sin(a+1)x+\sin x}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to0-}\frac{(a+1)\cos(a+1)x+\cos x}{1}$

$= a + 2$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}}$$

$$= \lim_{x\to 0+} \frac{\sqrt{1+bx}-1}{bx} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0+} \frac{\frac{b}{2\sqrt{1+bx}}-0}{b} = \frac{1}{2}$$

Since the function is continuous at $x = 0$, $a + 2 = c = \frac{1}{2}$. So, $a = -\frac{3}{2}$, $c = \frac{1}{2}$ and $b \in R$ **Ans. :** (D)

(3) A function f is continuous. If $f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right), \forall\, x \in D_f$ and $f(1) > 0$, then $\lim_{x\to 1} f(x) =$ ..... .

(A) 2 (B) 1 (C) 3 (D) limit does not exist

Sol. : Since the function f is continuous, $\lim_{x\to 1} f(x) = f(1)$

Taking $x = 1$ in the given equation, $(f(1))^2 = 2 f(1)$

$\therefore\ f(1) = 0$ or $f(1) = 2$. But $f(1) > 0$. Hence, $f(1) = 2$

$\lim_{x\to 1} f(x) = 2$ **Ans. :** (A)

(4) $$f(x) = \begin{cases} \dfrac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x - \cos x}; & x \neq 0 \\ k & ; x = 0 \end{cases}$$

If the function f is continous at $x = 0$, then k = ...

(A) 1 (B) 0 (C) 2 (D) –1

Sol. : If the function f is continous at $x = 0$, then

$$f(0) = k = \lim_{x\to 0} \frac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x - \cos x}$$

$$= \lim_{x\to 0} \frac{\log\left((1+x^2)^2 - x^2\right)}{1-\cos^2 x} \cos x$$

$$= \lim_{x\to 0} \frac{\log\left(1+x^2+x^4\right)}{\sin^2 x} \cos x$$

$$= \lim_{x\to 0} \frac{\log\left(1+(x^2+x^4)\right)}{x^2+x^4} \times \lim_{x\to 0} \frac{x^2}{\sin^2 x} \times \lim_{x\to 0} (1+x^2)\cos x$$

$$= 1$$

Ans. : (A)

(5) A non-zero function f satisfies $f(x + y) = f(x) f(y)$, $\forall\ x, y \in R$. If $f(2) = 9$, then $f(3) =$ .....

(A) 1 (B) 27 (C) 9 (D) 3

Sol. : The function which satisfies the given condition is $f(x) = a^x$, $a > 0$.

$f(2) = 9$. So, $a^2 = 9 \Rightarrow a = 3$

$f(x) = 3^x$ and hence $f(3) = 3^3 = 27$ **Ans. :** (B)

(6) For which value of $n \in N$, $\lim_{x\to 0} \frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero value ?

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $(\cos x - 1)(\cos x - e^x)$

$$= \left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)-1\right)\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\right]$$

$$= x^2\left(\frac{-1}{2}+\frac{x^2}{4!}+\ldots\right)\left(-x-x^2-\frac{x^3}{3!}-\ldots\right)$$

$$= x^3\left(\frac{-1}{2}+\frac{x^2}{4!}-\ldots\right)\left(-1-x-\frac{x^2}{3!}-\ldots\right)$$

$$\therefore \lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^3} = \frac{1}{2} \neq 0$$

$\therefore\ n = 3$ **Ans. :** (C)

Note : For $n = 1, 2$ the limit becomes zero and for $n > 3$, function is unbounded.

(7) Let $f(x) = g(x)\ \frac{e^{\frac{1}{x}}-e^{\frac{-1}{x}}}{e^{\frac{1}{x}}+e^{\frac{-1}{x}}}$ where, g is a continuous function. If $\lim_{x\to 0} f(x)$ exists, then...

(A) $g(x) = x + 2$ (B) $g(x) = x^2 + 4$

(C) $g(x) = xh(x)$ where $h(x)$ is a polynomial function (D) $g(x)$ is a non-zero constant function

Sol. : $\lim_{x\to 0+}\frac{e^{\frac{1}{x}}-e^{\frac{-1}{x}}}{e^{\frac{1}{x}}+e^{\frac{-1}{x}}} = \lim_{x\to 0+}\frac{1-e^{\frac{-2}{x}}}{1+e^{\frac{-2}{x}}} = 1$ $\left(e^{\frac{-2}{x}} \to 0\right)$

$\lim_{x\to 0-}\frac{e^{\frac{1}{x}}-e^{\frac{-1}{x}}}{e^{\frac{1}{x}}+e^{\frac{-1}{x}}} = \lim_{x\to 0-}\frac{e^{\frac{2}{x}}-1}{e^{\frac{2}{x}}+1} = -1$ $\left(e^{\frac{2}{x}} \to 0\right)$

$\therefore$ If $g(x) = xh(x)$, then $\lim_{x\to 0} f(x) = 0$ exists and is zero. **Ans. :** (C)

(8) $\lim_{\alpha\to\frac{\pi}{4}}\frac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}$ = ... (AIEEE : 2002)

(A) $\sqrt{2}$ (B) 1 (C) 2 (D) 0

Sol. : $\lim_{\alpha\to\frac{\pi}{4}}\frac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}\ \left(\frac{0}{0}\text{ form}\right) = \lim_{\alpha\to\frac{\pi}{4}}\frac{\cos\alpha+\sin\alpha}{1} = \sqrt{2}$ **Ans. :** (A)

(9) $\lim_{x\to 0}\frac{\sqrt{1-\cos 2\sqrt{x}}}{2\sqrt{x}}$ = ... (AIEEE : 2002)

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) limit does not exist

Sol. : If $x < 0$, then $\sqrt{x}$ and $\sqrt{1-\cos 2\sqrt{x}}$ are not defined.

$\therefore \lim\limits_{x\to 0-} \dfrac{\sqrt{1-\cos 2\sqrt{x}}}{2\sqrt{x}}$ does not exist. **Ans. :** (D)

(10) If $\lim\limits_{x\to 0} \dfrac{\log(3+x)-\log(3-x)}{x} = k$, then k = ... (AIEEE : 2003)

(A) $\dfrac{-1}{3}$ (B) $\dfrac{-2}{3}$ (C) $\dfrac{2}{3}$ (D) 0

Sol. : $\lim\limits_{x\to 0} \dfrac{\log(3+x)-\log(3-x)}{x} = k$ $\left(\dfrac{0}{0}\text{ form}\right)$

$\therefore \lim\limits_{x\to 0} \dfrac{\dfrac{1}{3+x}+\dfrac{1}{3-x}}{1} = k$

$\therefore \dfrac{1}{3} + \dfrac{1}{3} = k.$ So, $k = \dfrac{2}{3}$ **Ans. :** (C)

(11) If the quadratic equation $ax^2 + bx + c = 0$ has distinct real roots α, β, then

$\lim\limits_{x\to\alpha} \dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ = ...

(A) $\dfrac{b^2+4ac}{2}$ (B) 0 (C) $\dfrac{b^2-4ac}{2}$ (D) 1

Sol. : $\lim\limits_{x\to\alpha} \dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} = \lim\limits_{x\to\alpha} \dfrac{2\sin^2\left(\dfrac{ax^2+bx+c}{2}\right)}{(x-\alpha)^2}$

$= \lim\limits_{x\to\alpha} \dfrac{2\sin^2\left(\dfrac{a(x-\alpha)(x-\beta)}{2}\right)}{\left(\dfrac{a(x-\alpha)(x-\beta)}{2}\right)^2} \cdot \dfrac{a^2}{4}(x-\beta)^2$

$= \dfrac{a^2(\alpha-\beta)^2}{2} = \dfrac{a^2}{2}[(\alpha+\beta)^2 - 4\alpha\beta] = \dfrac{a^2}{2}\left[\dfrac{b^2}{a^2} - \dfrac{4c}{a}\right]$

$= \dfrac{b^2-4ac}{2}$ **Ans. :** (C)

Second method :

$\lim\limits_{x\to\alpha} \dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$

$= \dfrac{a\sin\big(a(x-\alpha)(x-\beta)\big)\big(x-\alpha+x-\beta\big)}{2(x-\alpha)a(x-\beta)} \times a(x-\beta) = \dfrac{a^2}{2}(\alpha-\beta)^2 = \dfrac{b^2-4ac}{2}$ **Ans. :** (C)

(12) $\lim\limits_{x\to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ = ... (JEE Main : 2015)

(A) 2 (B) $\frac{1}{2}$ (C) 4 (D) 3

Sol. : $\lim\limits_{x\to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$

$= \lim\limits_{x\to 0} \dfrac{2\sin^2 x(3+\cos x)}{x\dfrac{\tan 4x}{4x}\cdot 4x}$

$= \frac{1}{2} \lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^2 \dfrac{1}{\lim\limits_{x\to 0}\dfrac{\tan 4x}{4x}} \lim\limits_{x\to 0}(3+\cos x)$

$= \frac{1}{2}\cdot 1\cdot 1\cdot 4 = 2$ **Ans. :** (A)

(13) If a function f is differentiable from left and also from right at a, then f is ..... .

(A) continuous at a (B) not differentiable at a

(C) differentiable at 0 (D) not continuous at a

Sol. : $\lim\limits_{h\to 0+} (f(a) - f(a-h)) = \lim\limits_{h\to 0+} \dfrac{f(a)-f(a-h)}{h}\, h$

$= \lim\limits_{h\to 0+} f'(a^-)\cdot h = 0$ ($f'(a^-)$ exists)

Similarly, $\lim\limits_{h\to 0+} (f(a+h)) - f(a)) = 0$

$\therefore \lim\limits_{h\to 0+} f(a-h)) = \lim\limits_{h\to 0+} f(a+h) = f(a)$

$\therefore$ f is continuous at $x = a$. **Ans. :** (A)

(14) If $f(x) = (1+x)^n$, then $f(0) + f'(0) + \frac{1}{2} f''(0) + ... + \frac{1}{n!} f^n(0)$ = ...

(A) n (B) 2^n (C) 2^{n-1} (D) 1

Sol. : $f(0) = 1, f'(x) = n(1+x)^{n-1}, f''(x) = n(n-1)(1+x)^{n-2} ... f^{(n)}(x) = n(n-1)...1\,(1+x)^{n-n} = n!$

$\therefore f'(0) = n, f''(0) = n(n-1), ...f^n(0) = n!$

Required value $= 1 + n + \dfrac{n(n-1)}{2!} + ... + \dfrac{n!}{n!} = 2^n$ **Ans. :** (B)

(15) $\lim\limits_{n\to\infty} \dfrac{1+2^4+3^4+...+n^4}{n^5} - \lim\limits_{n\to\infty} \dfrac{1+2^3+3^3+...+n^3}{n^5}$ = ... (AIEEE : 2003)

(A) 0 (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{30}$

Sol. : Required limit $= \lim\limits_{n\to\infty} \dfrac{\sum n^4}{n^5} - \lim\limits_{n\to\infty} \dfrac{\sum n^3}{n^5}$

$= \lim\limits_{n\to\infty} \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30n^5} - \lim\limits_{n\to\infty} \dfrac{n^2(n+1)^2}{4\cdot n^5}$

$= \lim\limits_{n\to\infty} \dfrac{n^5\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(3+\frac{3}{n}-\frac{1}{n^2}\right)}{30n^5} - \lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^2}{4n} = \frac{1}{5} - 0 = \frac{1}{5}$ **Ans. :** (B)

(16) If $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$ is differentiable, then $k + m$ = ... (JEE Main : 2015)

(A) $\frac{10}{3}$ (B) 4 (C) 2 (D) $\frac{16}{5}$

Sol. : If $g(x)$ is differentiable, then g is continuous at $x = 3$.

$\therefore \lim_{x \to 3+} g(x) = g(3)$

$\therefore 3m + 2 = 2k$ (1)

$g(x)$ is differentiable. So, $g'(3+) = g'(3-)$

$\therefore m = k \frac{1}{2\sqrt{x+1}}$ $(x = 3)$

$\therefore m = \frac{k}{4}$. So, $k = 4m$ Hence, $3m + 2 = 8m$ (2)

$\therefore m = \frac{2}{5}, k = \frac{8}{5}$. So, $k + m = 2$ **Ans. :** (C)

(17) At how many points is the function $f(x) = |3 - |3 - |x|||$ not differentiable ?

(A) 2 (B) 3 (C) 5 (D) 6

Sol. : We know that $|x|$ is not differentiable at $x = 0$.

Given function f is not differentiable at $x = 0$, $|x| = 3$ and $|3 - |x|| = 3$ that is at $x = 0$, $x = \pm 3$, $x = \pm 6$ not differentiable.

Given function is not differentiable at 5 points. **Ans. :** (C)

(18) In which largest set is $f(x) = \frac{x}{1+|x|}$ differentiable ? (AIEEE : 2006)

(A) $(-\infty, 0) \cup (0, \infty)$ (B) $(-\infty, -1) \cup (-1, \infty)$ (C) $(-\infty, \infty)$ (D) $(0, \infty)$

Sol. : $f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \ge 0 \end{cases}$

$f'(x) = \begin{cases} \frac{1}{(1-x)^2}, & x < 0 \\ \frac{1}{(1+x)^2}, & x > 0 \end{cases}$ Here, $f(0) = 0$

$f'(0+) = \lim_{x \to 0+} \frac{\frac{x}{1+x}}{x} = 1$

$f'(0-) = \lim_{x \to 0-} \frac{\frac{x}{1-x}}{x} = 1$

$\therefore f'(0) = 1$

$\therefore f'(x)$ exists on $(-\infty, \infty)$. **Ans. :** (C)

(19) If $x^m y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ = ... (AIEEE : 2006)

(A) $\frac{y}{x}$ (B) $\frac{x+y}{xy}$ (C) xy (D) $\frac{x}{y}$

Sol. : $x^m y^n = (x + y)^{m+n}$ $\qquad x > 0,\ y > 0$

Apply *log* on both sides

$m \log x + n \log y = (m + n) \log (x + y)$

Differentiate,

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\therefore \frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\therefore \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right) = \frac{nx-my}{x(x+y)}$$

$\therefore$ If $nx \neq my$, then $\dfrac{dy}{dx} = \dfrac{y}{x}$

If $nx = my$, then also

$$m\frac{dy}{dx} = n$$

$$\therefore \frac{dy}{dx} = \frac{n}{m} = \frac{y}{x}$$ **Ans. :** (A)

(20) $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{3x+2}-2}{\sqrt[5]{x+30}-2}$ = ...

(A) 10 (B) 20 (C) 30 (D) 40

Sol. : $\lim\limits_{x\to 2} \dfrac{(3x+2)^{\frac{1}{3}}-2}{(x+30)^{\frac{1}{5}}-2}$ $\qquad \left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 2} \frac{\frac{1}{3}(3x+2)^{\frac{-2}{3}}\cdot 3}{\frac{1}{5}(x+30)^{\frac{-4}{5}}} = \frac{\frac{1}{4}.5}{\left(\frac{1}{16}\right)} = 20$$ **Ans. :** (B)

(21) $\lim\limits_{x\to\infty} x \sum\limits_{i=1}^{n}\left(e^{\frac{i}{x}}-1\right)$ = ...

(A) $\dfrac{n+1}{2}$ (B) $\dfrac{n^2+1}{2}$ (C) $\dfrac{n(n+1)}{2}$ (D) $\dfrac{n^2(n+1)^2}{4}$

Sol. : Required limit $= \lim\limits_{x\to\infty} x\left[\left(e^{\frac{1}{x}}-1\right)+\left(e^{\frac{2}{x}}-1\right)+\ldots+\left(e^{\frac{n}{x}}-1\right)\right]$

$$= \lim_{\frac{1}{x}\to 0}\left[\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}+2\frac{e^{\frac{2}{x}}-1}{\frac{2}{x}}+\ldots+n\frac{e^{\frac{n}{x}}-1}{\frac{n}{x}}\right] = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$ **Ans. :** (C)

(22) $\lim_{\theta\to 0} \frac{1-\cos\{1-\cos(1-\cos\theta)\}}{\theta^8}$ = ...

(A) $\frac{1}{4}$ (B) $\frac{1}{16}$ (C) $\frac{1}{64}$ (D) $\frac{1}{128}$

Sol. : We note that, $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ (1)

Now, required limit = $\lim_{\theta\to 0} \frac{1-\cos\{1-\cos(1-\cos\theta)\}}{\{1-\cos(1-\cos\theta)\}^2} \times \frac{\{1-\cos(1-\cos\theta)\}^2}{(1-\cos\theta)^4} \times \frac{(1-\cos\theta)^4}{\theta^8}$

As $\theta \to 0$, $1 - \cos\theta \to 0$ and $1 - \cos(1 - \cos\theta) \to 0$

Using (1), required limit = $\frac{1}{2}\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4 = \frac{1}{128}$ **Ans. :** (D)

(23) $\lim_{n\to\infty} \cos\frac{x}{2} \cos\frac{x}{4} \cos\frac{x}{8} \ldots \cos\frac{x}{2^n}$ = ...

(A) $\sin x$ (B) $\frac{\sin x}{x}$ (C) $\frac{\cos x}{x}$ (D) $\cos x$

Sol. : $\lim_{n\to\infty} \cos\frac{x}{2} \cos\frac{x}{4} \cos\frac{x}{8} \ldots \cos\frac{x}{2^n}$

$= \lim_{n\to\infty} \frac{2^n \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots\cos\frac{x}{2^n}\sin\frac{x}{2^n}}{2^n \sin\frac{x}{2^n}} = \lim_{n\to\infty} \frac{\sin x}{x} \frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}} = \frac{\sin x}{x}$ **Ans. :** (B)

(24) For $x\sqrt{1+y} + y\sqrt{1+x} = 0$, $\frac{dy}{dx}$ = ... $(x \neq 0, -1)$

(A) $\frac{1}{(1+x)^2}$ (B) $\frac{-1}{(1+x)^2}$ (C) $\frac{1}{1+x^2}$ (D) $\frac{-1}{1+x^2}$

Sol. : $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\therefore x^2(1+y) = y^2(1+x)$

$\therefore x^2 - y^2 = xy(y - x)$

$\therefore x + y + xy = 0$. So, $y = \frac{-x}{1+x} = -1 + \frac{1}{1+x}$. Hence, $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$ **Ans. :** (B)

Note : If $x = y$, then $x\sqrt{1+x} = 0$ but $x \neq 0, -1$.

(25) If $\tan\frac{y}{2} = \sqrt{\frac{1-e}{1+e}} \tan\frac{x}{2}$, then $\frac{dy}{dx}$ = ...

(A) $\frac{\sqrt{1-e^2}}{1+e\cos x}$ (B) $\sqrt{\frac{1-e^2}{1+e\cos x}}$ (C) $\frac{\sqrt{1+e^2}}{1+e\cos x}$ (D) $\frac{\sqrt{1-e^2}}{1-e\cos x}$

Sol. : $\tan\frac{y}{2} = \sqrt{\frac{1-e}{1+e}} \tan\frac{x}{2}$ (1)

Differentiate w.r.t. 'x',

$\sec^2\frac{y}{2} \cdot \frac{1}{2}\frac{dy}{dx} = \sqrt{\frac{1-e}{1+e}} \sec^2\frac{x}{2} \cdot \frac{1}{2}$

$$\left[1+\tan^2\frac{y}{2}\right]\frac{dy}{dx} = \sqrt{\frac{1-e}{1+e}}\ \sec^2\frac{x}{2}$$

$$\therefore \left[1+\frac{1-e}{1+e}\tan^2\frac{x}{2}\right]\frac{dy}{dx} = \sqrt{\frac{1-e}{1+e}}\ \sec^2\frac{x}{2} \qquad \text{(From (1))}$$

Multiplying both sides by $(1+e)\cos^2\frac{x}{2}$,

$$\left[(1+e)\cos^2\frac{x}{2} + (1-e)\sin^2\frac{x}{2}\right]\frac{dy}{dx} = \sqrt{1-e^2}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{1-e^2}}{1+e\cos x} \qquad \textbf{Ans. :} \text{ (A)}$$

(26) If $y = \frac{1}{\sqrt{\cos 2x}}$, then $y_2 + y =$ .....

(A) y^5 (B) $2y^5$ (C) $3y^5$ (D) $4y^5$

Sol. : Here, $y^2\cos 2x = 1$ (1)

$\therefore \cos 2x\ 2yy_1 - 2\sin 2x \cdot y^2 = 0$

$\therefore y_1 = y\tan 2x$

$\therefore y_2 = y_1\tan 2x + 2\sec^2 2x \cdot y$

$= y\tan^2 2x + 2\sec^2 2x\ y$

$= 3\sec^2 2x \cdot y - y = 3y^4 \cdot y - y$ (From (1))

$\therefore y_2 + y = 3y^5$ **Ans. :** (C)

(27) If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then $(x^2+4)\left(\frac{dy}{dx}\right)^2 =$ ..... (y^2+4)

(A) n (B) n^2 (C) $2n$ (D) n^3

Sol. : $x^2 = \sec^2\theta - 2 + \cos^2\theta$ and $y^2 = \sec^{2n}\theta - 2 + \cos^{2n}\theta$

$\therefore x^2 + 4 = (\sec\theta + \cos\theta)^2$, $y^2 + 4 = (\sec^n\theta + \cos^n\theta)^2$

Now, $x = \sec\theta - \cos\theta$

$$\therefore \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \sin\theta\sec\theta(\sec\theta + \cos\theta)$$

$$\therefore \frac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$$

$$= n\sin\theta\sec\theta(\sec^n\theta + \cos^n\theta)$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2} = \frac{n^2(y^2+4)}{(x^2+4)}$$

$$\therefore (x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4) \qquad \textbf{Ans. :} \text{ (B)}$$

(28) If $f(x) = p\,|\sin x| + qe^{|x|} + r|x|^3$ is differentiable at $x = 0$, then ..... $\left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$

(A) $q + r = 0, p \in R$ (B) $p + q = 0, p \in R$

(C) $q = 0, r = 0, p \in R$ (D) $r = 0, p = 0, q \in R$

Sol. : If $x < 0$, then $f(x) = -p \sin x + qe^{-x} - rx^3$ $-\frac{\pi}{2} < x < 0$

$f'(x) = -p \cos x - qe^{-x} - 3r x^2$

$\therefore\ f'(0-) = -p - q$

If $x > 0$, then $f(x) = p \sin x + qe^{x} + rx^3$ $0 < x < \frac{\pi}{2}$

$\therefore f'(x) = p \cos x + qe^{x} + 3r x^2$ $x > 0$

$\therefore\ f'(0+) = p + q$

If f is differentiable at $x = 0$, then

$-p - q = p + q$. So, $2(p + q) = 0$. Hence, $p + q = 0$ **Ans. :** (B)

(29) If $y = \tan^{-1}\left(\frac{\log e\, x^{-2}}{\log e\, x^2}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$, then $\frac{dy}{dx}$ = ...

(A) 2 (B) 1 (C) 0 (D) x

Sol. : $y = \tan^{-1}\left(\frac{\log e - 2\log x}{\log e + 2\log x}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$

$= \tan^{-1}\left(\frac{1 - 2\log x}{1 + 2\log x}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$

$= \frac{\pi}{4} - \tan^{-1}(2 \log x) + \tan^{-1} 3 + \tan^{-1}(2 \log x)$

$y = \frac{\pi}{4} + \tan^{-1} 3$ So, $\frac{dy}{dx} = 0$ **Ans. :** (C)

(Here, $\log x > 0$ and $0 < 6\log x < 1$. Answer will remain same otherwise also.)

(30) If $3f(x) - 2f\left(\frac{1}{x}\right) = x$, then $f'(2)$ = ...

(A) $\frac{2}{7}$ (B) $\frac{1}{2}$ (C) 2 (D) $\frac{7}{2}$

Sol. : $3f(x) - 2f\left(\frac{1}{x}\right) = x$

$\therefore\ 3f'(x) - 2f'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = 1$

Substituting $x = 2$, we get, $3f'(2) + \frac{2}{4} f'\left(\frac{1}{2}\right) = 1$ (1)

Substituting $x = \frac{1}{2}$, we get, $3f'\left(\frac{1}{2}\right) + 8f'(2) = 1$ (2)

Solving equations (1) and (2), we get, $f'(2) = \frac{1}{2}$ **Ans. :** (B)

(31) $f(x) = \dfrac{tan[e^2]x^3 - tan[-e^2]x^3}{sin^3 x}$, $x \neq 0$. If f is continuous at $x = 0$, then $f(0)$ = ...

(A) 15 (B) 12 (C) –12 (D) 14

Sol. : $7 < e^2 < 8$. So, $[e^2] = 7$. Hence $[-e^2] = -8$

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{tan 7x^3 - tan(-8)x^3}{sin^3 x}$$

$$= \lim_{x\to 0} \frac{tan 7x^3}{7x^3} \cdot 7 + \lim_{x\to 0} \frac{tan 8x^3}{8x^3} \cdot 8 \qquad \left(\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3 = 1\right)$$

$= 7 + 8 = 15$ **Ans. :** (A)

(32) At how many points is $f(x) = \dfrac{1}{\log|x|}$ discontinuous ?

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : f is not defined at $x = 0, \pm 1$. At other points f is continuous. **Ans. :** (C)

(33) $f(x) = \dfrac{1}{1-x}$, $g(x) = f^{3n}(x)$ where $f^n(x) = fofof...of$ (n times). At how many points is function g discontinous ?

(A) 2 (B) $2n$ (C) $3n$ (D) $2n + 1$

Sol. : $f(x) = \dfrac{1}{1-x}$ So, f is discontinuous at $x = 1$.

$(fof)(x) = \dfrac{x-1}{x}$ So, fof is discontiunous at $x = 0$.

$$(fofof)(x) = \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}} = x$$

$g(x) = f^{3n}(x) = x$

$\therefore$ g is discontinuous at 0 and 1 only. **Ans. :** (A)

(34) f and g are differentiable functions. If $g'(a) = 2$, $g(a) = b$ and $fog = I$, then $f'(b)$ = ...

(A) $\dfrac{1}{2}$ (B) 2 (C) $\dfrac{2}{3}$ (D) 1

Sol. : $f(g(x)) = x$

$\therefore f'(g(x))\, g'(x) = 1$

$\therefore f'(g(a))\, g'(a) = 1$

$\therefore f'(b) \cdot 2 = 1$

$\therefore f'(b) = \dfrac{1}{2}$ **Ans. :** (A)

(35) $y = x^n (a\ cos\ (log\ x) + b\ sin\ (\log x))$. If, y satisfies $x^2y_2 + (1 - 2n)\, xy_1 + Ay = 0$, then A = ...

(A) n (B) $1 + n^2$ (C) $1 + \dfrac{1}{n}$ (D) $1 - n^2$

Sol. : $y = x^n\ (a\ cos\ (\log x) + b\ sin\ (\log x))$

$y_1 = nx^{n-1}\ (a\ cos\ (\log x) + b\ sin\ (\log x)) + x^n \left(-asin(\log x)\frac{1}{x} + (b\,cos(\log x)\frac{1}{x}\right)$

$\therefore\ xy_1 = ny + x^n\ (-\ a\ sin\ (\log x) +\ b\ cos\ (\log x))$

$\therefore\ xy_2 + y_1 = ny_1 + nx^{n-1}(-\ a\ sin\ (\log x) + b\ cos\ (\log x)) + x^n(-a\ cos(\log x)\frac{1}{x} - b\ sin(\log x)\ \frac{1}{x})$

$\therefore\ x^2y_2 + xy_1 = nxy_1 + n\ (xy_1 - ny) - y$

$\therefore\ x^2y_2 + (1 - 2n)\ xy_1 + (1 + n^2)y = 0$ **Ans. :** (B)

(36) If $x = f(t)$, $y = g(t)$, then $\frac{d^2y}{dx^2}$ = ...

(A) $\frac{f'g'' - g'f''}{(f')^2}$ (B) $\frac{f'g'' - g'f''}{(f')^3}$ (C) $\frac{f''}{g''}$ (D) $\frac{g''}{f''}$

Sol. : $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{g''(t)\,f'(t) - g'(t)\,f''(t)}{[f'(t)]^2}\ \frac{1}{f'(t)} = \frac{f'g'' - g'f''}{(f')^3}$ **Ans. :** (B)

(37) If $f(x) = \begin{cases} (1+|\ sin\ x\ |)^{\frac{a}{|sin\,x|}}, & \frac{-\pi}{6} < x < 0 \\ b & x = 0 \\ e^{tan2x/tan3x} & 0 < x < \frac{\pi}{6} \end{cases}$ is continuous on $\left(\frac{-\pi}{6}, \frac{\pi}{6}\right)$, then .....

(A) $a = \frac{2}{3}$, $b = e^2$ (B) $a = \frac{1}{3}$, $b = e^{\frac{1}{3}}$ (C) $a = \frac{2}{3}$, $b = e^{\frac{2}{3}}$ (D) $a = e^{\frac{2}{3}}$, $b = e^{\frac{2}{3}}$

Sol. : $b = f(0) = \lim_{x\to 0+} f(x) = \lim_{x\to 0} e^{\frac{tan2x}{tan3x}} = e^{\frac{2}{3}}$

Moreover, $b = f(0) = \lim_{x\to 0-} f(x) = \lim_{x\to 0} \left(1+|sin\,x|\right)^{\frac{1}{|sin\,x|}a} = e^a = e^{\frac{2}{3}}$ So, $a = \frac{2}{3}$ **Ans. :** (C)

(38) If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$, then $\frac{y'}{y}$ = ...

(A) $\frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$ (B) $\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}$ (C) $\frac{1}{x}$ (D) $\frac{1}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$

Sol. : $y = \frac{ax^2 + bx(x-a) + c(x-a)(x-b) + (x-a)(x-b)(x-c)}{(x-a)(x-b)(x-c)}$

$$y = \frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\log y = 3\log x - \log (x-a) - \log (x-b) - \log (x-c)$$

$$\frac{y'}{y} = \frac{3}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}$$

$$= \frac{1}{x}\left[3 - \frac{x}{x-a} - \frac{x}{x-b} - \frac{x}{x-c}\right]$$

$$= \frac{1}{x}\left[3 - \frac{x-a+a}{x-a} - \frac{x-b+b}{x-b} - \frac{x-c+c}{x-c}\right]$$

$$= \frac{1}{x}\left[\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right]$$ **Ans. :** (A)

(39) If a function f is continuous on R and $f\left(\frac{1}{4n}\right) = (\sin e^n)\, e^{-n^2} + \frac{n^2}{n^2+1}$, then the value of $f(0)$ is ..... .

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) –1

Sol. : The function f is continuous at $x = 0$.

$$\therefore f(0) = \lim_{n\to\infty} f\left(\frac{1}{4n}\right) = \lim_{n\to\infty}\left[(\sin e^n)e^{-n^2} + \frac{n^2}{n^2+1}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(\sin e^n)}{e^{n^2}} + \frac{1}{1+\frac{1}{n^2}}\right] = 1$$ **Ans. :** (A)

(40) If $f(x) = \frac{x}{2} - 1$, $x \in$ R on the interval $[0, \pi]$, then

(A) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both continuous. (B) $\tan(f(x))$ and $\frac{1}{f(x)}$ are both discontinuous.

(C) $\tan(f(x))$ and $f^{-1}(x)$ are both continuous. (D) $\tan(f(x))$ and $f^{-1}(x)$ are both discontinuous.

Sol. : $f(x) = \frac{x}{2} - 1$ is continuous on $[0, \pi]$.

$y = \tan f(x)$ is continuous on $[0, \pi]$ as $f(x) \in \left[-1, \frac{\pi}{2} - 1\right]$.

$\therefore$ $\tan f(x)$ is continuous on $[0, \pi]$ and since $\frac{1}{f(x)}$ is not defined at $x = 2$, it is discontinuous on $[0, \pi]$.

$\therefore$ $f^{-1}(x) = 2(x + 1)$ is continuous on $[0, \pi]$.

$\therefore$ $\tan f(x)$ and $f^{-1}(x)$ both are continuous on $[0, \pi]$. **Ans. :** (C)

(41) If $f(x) = \left\{\frac{\log(1+x)^{1+x}}{x^2} - \frac{1}{x}\right\}$, then $\lim_{x\to 0} f(x)$ = ...

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Sol. : $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(1+x)\log(1+x) - x}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0} \frac{\log(1+x)+1-1}{2x}$

$= \frac{1}{2} \lim_{x\to 0} \frac{\log(1+x)}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \frac{1}{2} \lim_{x\to 0} \frac{1}{1+x} = \frac{1}{2}$ **Ans. :** (B)

(42) If $f(x) = x \frac{e^{[x]+|x|} - 2}{[x]+|x|}$, then $\lim_{x\to 0} f(x)$ = ...

(A) 1 (B) 0 (C) –1 (D) limit does not exist

Sol. : For $-1 < x < 0$, $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x \frac{e^{-1-x} - 2}{-1-x} = 0$

For $0 < x < 1$, $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x \frac{e^x - 2}{x} = \lim_{x\to 0+} e^x - 2 = -1$

$\therefore \lim_{x\to 0} f(x)$ does not exists. **Ans. :** (D)

(43) $\lim_{x\to 0} x^7 \left[\frac{1}{x^3}\right]$ = ...

(A) 0 (B) 1 (C) –1 (D) limit does not exist

Sol. : We know that, $x - 1 < [x] \le x, \ \forall\, x \in R$

$\therefore \frac{1}{x^3} - 1 < \left[\frac{1}{x^3}\right] \le \frac{1}{x^3}$

$\therefore x^7\left(\frac{1}{x^3} - 1\right) < x^7 \left[\frac{1}{x^3}\right] \le x^4, \ \forall x \ \lim_{x\to 0} x^7 \left[\frac{1}{x^3} - 1\right] = \lim_{x\to 0} \frac{x^7(1-x^3)}{x^3} = 0$

$\therefore$ by using Sandwich theorem, $\lim_{x\to 0} x^7 \left[\frac{1}{x^3}\right] = 0$ **Ans. :** (A)

(44) A non-zero function f satisfies $f(x-y) = \frac{f(x)}{f(y)}$, $\forall x, y \in R$. If $f'(0) = 5$ and $f'(5) = 5$, then $f(5)$ = ...

(A) 1 (B) 5 (C) 2 (D) 0

Sol. : From the given functional relation, $f(0) = 1$ and $f(-x) = \frac{1}{f(x)}$ (Taking $x = y$ and $x = 0, y = x$ respectively)

Now, $5 = f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$

$$= \lim_{h\to 0} \frac{f(h-(-5))-f(5)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(h)}{f(-5)}-f(5)}{h}$$

$$= \lim_{h\to 0} \frac{f(5)\,f(h)-f(5)}{h} \qquad \left(f(-x)=\frac{1}{f(x)}\right)$$

$\therefore \quad 5 = f(5)\,f(0) = 5f(5) \qquad \left(\frac{f(h)-1}{h}=\frac{f(h)-f(0)}{h}\right)$

$\therefore \quad 5 = 5\,f(5).$ So, $f(5) = 1$ **Ans. :** (A)

(45) If $f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, & x<\frac{\pi}{2} \\ a, & x=\frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2}, & x>\frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then $\frac{b}{a}$ = ...

(A) 2 (B) 4 (C) 6 (D) 8

Sol. : $\lim\limits_{x\to\frac{\pi}{2}-} f(x) = \lim\limits_{x\to\frac{\pi}{2}-} \frac{1-\sin^3 x}{3\cos^2 x}$

$$= \lim_{x\to\frac{\pi}{2}-} \frac{(1-\sin x)\,(1+\sin x+\sin^2 x)}{3(1-\sin x)(1+\sin x)} = \frac{1}{2}$$

$$\lim_{x\to\frac{\pi}{2}+} f(x) = \lim_{x\to\frac{\pi}{2}+} \frac{b(1-\sin x)}{(\pi-2x)^2}$$

$$= \lim_{x\to\frac{\pi}{2}+} \frac{b\cos^2 x}{4\left(\frac{\pi}{2}-x\right)^2}\times\frac{1}{1+\sin x}$$

$$= \lim_{x\to\frac{\pi}{2}+} \frac{b\sin^2\left(\frac{\pi}{2}-x\right)}{4\left(\frac{\pi}{2}-x\right)^2} \times \lim_{x\to\frac{\pi}{2}+} \frac{1}{1+\sin x} = \frac{b}{8}$$

Since f is continuous at $x = \frac{\pi}{2}$, we must have $\frac{1}{2} = a = \frac{b}{8}$. So, $\frac{b}{a} = 8$ **Ans. :** (D)

(46) $\lim\limits_{x\to 0} \left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$ = ...

(A) e (B) e^2 (C) e^{-2} (D) $\frac{1}{e}$

Sol. : $\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$

$= \lim_{x\to 0}\left[\frac{1+\tan x}{1-\tan x}\right]^{\frac{1}{x}}$

$= \frac{\lim_{x\to 0}\left[(1+\tan x)^{\frac{1}{\tan x}}\right]^{\frac{\tan x}{x}}}{\lim_{x\to 0}\left[(1-\tan x)^{-\frac{1}{\tan x}}\right]^{\frac{-\tan x}{x}}} = \frac{e^1}{e^{-1}} = e^2$ **Ans. :** (B)

(47) If $f(x) = \begin{cases} |x|, & x \le 2 \\ [x], & x > 2 \end{cases}$, then .....

(A) $\lim_{x\to 2-} f(x) = -2$ (B) $\lim_{x\to 2+} f(x) = -2$

(C) $\lim_{x\to 2+} f(x) = f(2)$ (D) $\lim_{x\to 2} f(x)$ does not exist

Sol. : $\lim_{x\to 2+} f(x) = \lim_{x\to 2+} 2 = f(2)$ because $f(2) = |2| = 2$ **Ans. :** (C)

(48) If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & x \neq 0 \\ 0 \quad ; & x = 0 \end{cases}$, then f is ..... at $x = 0$.

(A) differentiable (B) not differentiable but continuous

(C) neither continuous nor differentiable (D) not defined

Sol. : $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} xe^{\frac{-2}{x}} = 0$

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x = 0$

$f(0) = 0$. So, $\lim_{x\to 0} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$.

$f'(0-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h}{-h} = 1$

$f'(0+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{he^{\frac{-2}{h}}}{h} = 0$

$\therefore$ f is not differentiable at $x = 0$. **Ans. :** (B)

(49) If $f(x) = \frac{2^x - 1}{\sqrt{1+ax}-1}$ and $\lim_{x\to 0} f(x) = \log_e 4$, then a = ...

(A) $\log_e 2$ (B) 1 (C) $\log_e 2 - 1$ (D) $\log_e 2 + 1$

Sol. : $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{2^x-1}{\sqrt{1+ax}-1} = \lim_{x\to 0} \dfrac{\frac{2^x-1}{x}}{\frac{\sqrt{1+ax}-1}{x}}$

$= \lim_{x\to 0} \dfrac{2^x-1}{x} \times \lim_{x\to 0} \dfrac{\sqrt{1+ax}+1}{\frac{1+ax-1}{x}}$

$= \dfrac{2}{a} \log 2 = \dfrac{1}{a} \log 4.$ So, $\dfrac{1}{a} \log_e 4 = \log_e 4.$ Hence, $a = 1$

Ans. : (B)

(50) If $f'(2) = 1$, then $\lim_{h\to 0} \dfrac{f(2+h^2)-f(2-h^2)}{2h^2}$ = ...

(A) 0 (B) 1 (C) 2 (D) $\frac{1}{2}$

Sol. : $\lim_{h\to 0} \dfrac{f(2+h^2)-f(2-h^2)}{2h^2} = \lim_{h\to 0} \dfrac{f'(2+h^2)2h - f'(2-h^2)(-2h)}{4h}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{h\to 0} \dfrac{f'(2+h^2)+f'(2-h^2)}{2} = f'(2) = 1$ **Ans. :** (B)

(51) Let f be twice differentiable on $[0, 1]$ and $|f''(x)| \le 1$, $\forall\, x \in [0, 1]$. If $f(0) = f(1)$, then on $(0, 1)$

(A) $|f(x)| > 1$ (B) $|f(x)| < 1$ (C) $|f'(x)| > 1$ (D) $|f'(x)| < 1$

Sol. : Since $f(0) = f(1)$, by Rolle's theorem there exists c, $0 < c < 1$, such that $f'(c) = 0$

Let if possible there be another point $x \in (0, 1)$ having this property ($x \ne c$). Using Lagrange's Mean value theorem in (x, c), for f' we have

$f'(x) - f'(c) = (x - c) f''(c_1)$ $(x < c_1 < c)$

$\therefore |f'(x)| = |x - c|\ |f''(c_1)| < 1$ $(|x - c| < 1\ |, f''(c_1)| \le 1)$

$\therefore |f'(x)| < 1$ **Ans. :** (D)

Note : Also the same is true in the interval (c, x).

(52) Which of the following function on $[-1, 1]$ satisfies the conditions of Rolle's theorem ?

(A) $f(x) = |x|[x]$ (B) $f(x) = [x] + [-x]$ (C) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ (D) $f(x) = |x| - |\sin x|$

Sol. : (a) $f(x) = |x|[x] = \begin{cases} x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

$\therefore f$ is not differentiable at $x = 0$.

(b) $f(x) = [x] + [-x] = \begin{cases} 0, & x \in Z \\ -1, & x \notin Z \end{cases}$

$\therefore f$ is not continuous at $x = 0$.

(c) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\sin x}{x} = 1$ and $f(0) = 0$

$\therefore f$ is not continuous at $x = 0$.

(d) $f(x) = \begin{cases} x - \sin x & 0 \le x \le 1 \\ -x + \sin x & -1 \le x < 0 \end{cases}$

$\therefore$ f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

Moreover, $f(-1) = f(1)$

Thus, conditions of Roll's theorem are satisfied. **Ans. :** (D)

(53) If $y = (1 + x)(1 + x^2)(1 + x^4)... (1 + x^{2n})$, then the value of $\frac{dy}{dx}$ at $x = 0$ is .....

(A) 0 (B) -1 (C) 1 (D) 2

Sol. : $y = (1 + x)(1 + x^2)(1 + x^4)... (1 + x^{2n})$

$\therefore \log y = \log(1 + x) + \log(1 + x^2) + \log(1 + x^4) + ... + \log(1 + x^{2n})$

Differentiating,

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + ... + \frac{2n\,x^{2n-1}}{(1+x)^{2n}}\right]$$

$$\therefore \frac{dy}{dx} = y\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + ... + \frac{2n\,x^{2n-1}}{(1+x)^{2n}}\right]. \quad \text{So, } \left(\frac{dy}{dx}\right)_{x=0} = y(0) \cdot 1 = 1$$

Second method :

$y(1 - x) = (1 - x)(1 + x)... (1 + x^{2n}) = 1 - x^{4n}$

$$\frac{dy}{dx} = \frac{(1-x)(-4n\,x^{4n-1}) + (1-x^{4n})}{(1-x)^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=0} = 1$$ **Ans. :** (C)

(54) Let $f : R \to R$, $f(x + y) = f(x) + f(y)$; $\forall\, x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then

(A) $f(x)$ is differentiable on (a, b) where $a < 0 < b$.

(B) $f'(x)$ is constant

(C) f is not continuous at R

(D) $f(x)$ is not differentiable at some points. (IIT : 2011)

Sol. : $f(x + y) = f(x) + f(y)$

Putting $x = y = 0$, we get $f(0) = f(0) + f(0)$. So, $f(0) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x+0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x) + f(h) - f(x) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

$\therefore f'(x) = \lambda$, where λ is constant.

$\therefore f(x) = \lambda x + c$, where c is constant

Moreover, $f(0) = 0$. So, $c = 0$

$\therefore f(x) = \lambda x$ **Ans. :** (B)

(55) If $\lim_{x\to 0} [1+x\log(1+b^2)]^{\frac{1}{x}} = 2\ b\ sin^2\ \theta$, $b > 0$ and $\theta \in [-\pi, \pi]$, then θ = ...

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

Sol. : $\lim_{x\to 0}\left[[1+x\log(1+b^2)]^{\frac{1}{x\log(1+b^2)}}\right]^{\log(1+b^2)} = e^{\log(1+b^2)} = 1 + b^2$

$\therefore\ 1 + b^2 = 2b\ sin^2\ \theta$

$\therefore\ sin^2\ \theta = \frac{1+b^2}{2b} \geq 1$. But $sin^2\ \theta \leq 1$

$\therefore\ sin^2\ \theta = 1$ $(1 + b^2 \geq 2b)$

$\therefore\ \theta = \pm\frac{\pi}{2}$ **Ans. :** (D)

(56) The left hand derivative of $f(x) = [x]\ sin\ \pi\ x$, at $x = k$, $k \in Z$ is ..... .

(A) $(-1)^k\ (k-1)\pi$ (B) $(-1)^{k-1}\ (k-1)\pi$ (C) $(-1)^k\ k\pi$ (D) $(-1)^{k-1}\ k\pi$

Sol. : $f'(k-) = \lim_{h\to 0} \frac{f(k) - f(k-h)}{h}$

$= \lim_{h\to 0} \frac{[k]\sin k\pi - [k-h]\sin(k-h)\pi}{h}$

$= \lim_{h\to 0} -(k-1)\ \frac{\sin(k-h)\pi}{h}$ $\left(\frac{0}{0}\text{ form}\right)$ $(sin\ k\pi = 0,\ [k-h] = k-1)$

$= -(k-1)\ \lim_{h\to 0} (-\pi)\ cos\ (k-h)\pi$

$= (k-1)\ \pi\ (-1)^k$ **Ans. :** (A)

(57) If $f(x) = \begin{cases} -x-\frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -cos\ x, & -\frac{\pi}{2} < x \leq 0 \\ x-1, & 0 < x \leq 1 \\ \log x, & x > 1 \end{cases}$, then

(A) $f(x)$ is continuous at $x = -\frac{\pi}{2}$ (B) $f(x)$ is not differentiable at $x = 0$

(C) $f(x)$ is differentiable at $x = -\frac{3}{2}, 1$ (D) (A), (B), (C) are true.

Sol. : $f\left(-\frac{\pi}{2}\right) = 0$

$\lim_{x\to -\frac{\pi}{2}-} f(x) = \lim_{x\to -\frac{\pi}{2}-} \left(-x-\frac{\pi}{2}\right) = 0$

$\lim_{x\to -\frac{\pi}{2}+} f(x) = \lim_{x\to -\frac{\pi}{2}+} (-cos\ x) = 0.$

$\therefore \lim_{x \to -\frac{\pi}{2}} f(x) = \lim_{x \to -\frac{\pi}{2}+} f(x) = f\left(-\frac{\pi}{2}\right) = 0$

$\therefore f$ is continuous at $x = -\frac{\pi}{2}$. Hence (A) is true.

Moreover, $f'(0+) = 1, f'(0-) = 0$. So, (B) is true.

$f'(1+) = 1,\ f'(1-) = 1$

$\therefore f$ is differentiable at $x = 1$.

Here, $-\frac{3}{2} > -\frac{\pi}{2}$.

$\therefore f(x) = -\cos x$. So, f is differentiable at $-\frac{3}{2}$.

$\therefore$ (C) is true. **Ans. :** (D)

(58) If $y = \cot^{-1}\left(\frac{\log\left(\frac{e}{x^2}\right)}{\log(ex^2)}\right) + \cot^{-1}\left(\frac{\log(ex^4)}{\log\left(\frac{e^2}{x^2}\right)}\right)$, then $\frac{dy}{dx}$ = ... $0 < \log x < \frac{1}{2}$

(A) -1 (B) 0 (C) 1 (D) 2

Sol. : $y = \cot^{-1}\left(\frac{\log\left(\frac{e}{x^2}\right)}{\log(ex^2)}\right) + \cot^{-1}\left(\frac{\log(ex^4)}{\log\left(\frac{e^2}{x^2}\right)}\right)$

$= \tan^{-1}\left(\frac{1+\log x^2}{1-\log x^2}\right) + \tan^{-1}\left(\frac{2-\log x^2}{1+2\log x^2}\right)$

$= \tan^{-1} 1 + \tan^{-1}(\log x^2) + \tan^{-1} 2 - \tan^{-1}(\log x^2)$

$= \tan^{-1} 1 + \tan^{-1} 2$

$\therefore \frac{dy}{dx} = 0$

[If $\log x^2 > 1$, then also by adding constant, $\frac{dy}{dx} = 0$] **Ans. :** (B)

(59) If $f(x) = (x+1)(x+2)(x+3) \ldots (x+100)$ and $g(x) = f(x) f''(x) \ldots (f'(x))^2$, then $g(x) = 0$ has

(A) no solution (B) one solution

(C) two solutions (D) at least three solutions.

Sol. : $\log f(x) = \log(x+1) + \log(x+2) + \ldots + \log(x+100)$

$\therefore \frac{f'(x)}{f(x)} = \frac{1}{x+1} + \frac{1}{x+2} + \ldots + \frac{1}{x+100}$

By differentiating,

$\frac{f(x) f''(x) - (f'(x))^2}{(f(x))^2} = \frac{-1}{(x+1)^2} - \frac{1}{(x+2)^2} - \cdots - \frac{1}{(x+100)^2} < 0$

$\therefore\ f(x)\,f''(x) - (f'(x))^2 = g(x) < 0$

$\therefore\ g(x) = 0$ has no solution. **Ans. :** (A)

(60) $f(x) = g'(x)\ \dfrac{e^{\frac{a}{x}} - e^{-\frac{a}{x}}}{e^{\frac{a}{x}} + e^{-\frac{a}{x}}}$ where g' is derivative of g and is continuous and $a > 0$. If $\lim\limits_{x\to 0} f(x)$ exists, then .....

(A) $g(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^{-n}$ where $a_i \in R,\ i = 1, 2 ...n$ and $a_1 \neq 0\ a_n \neq 0$

(B) g is a function, such that $g'(0) = 0$ (C) $g(x) = \sin x$ (D) $g(x) = \log(1 + x)$

Sol. : $\lim\limits_{x\to 0+} \dfrac{e^{\frac{a}{x}} - e^{-\frac{a}{x}}}{e^{\frac{a}{x}} + e^{-\frac{a}{x}}} = \lim\limits_{x\to 0+} \dfrac{1 - e^{\frac{-2a}{x}}}{1 + e^{\frac{-2a}{x}}} = 1$

and $\lim\limits_{x\to 0-} \dfrac{e^{\frac{a}{x}} - e^{\frac{-a}{x}}}{e^{\frac{a}{x}} + e^{\frac{-a}{x}}} = \lim\limits_{x\to 0-} \dfrac{e^{\frac{2a}{x}} - 1}{e^{\frac{2a}{x}} + 1} = -1$

If $g'(0) = 0$, then only $\lim\limits_{x\to 0} f(x)$ exists. **Ans. :** (B)

(61) Let a function f be real and non-negative. If $f'(x) \leq p\,f(x)$ for $x \geq 1$, where $p > 0$ and $f(1) = 0$, then $[f(\sqrt{e}) + f(\sqrt{\pi})]$

(A) 0 (B) R^+ (C) R^- (D) $\notin R$

Sol. : $(f'(x) - p\,f(x)) \leq 0$

$\therefore\ \dfrac{d}{dx}\ (e^{-px} f(x)) \leq 0$

Let, $g(x) = e^{-px} f(x)$

Now, $g'(x) \leq 0,\ \forall\ x \geq 1$. So, g is decreasing.

$g(x) \leq g(1) \qquad \forall\ x \geq 1$

$\therefore\ g(x) \leq 0.$ So, $e^{-px} f(x) \leq 0.$ Hence, $f(x) \leq 0,\ \forall\ x \geq 1$

But $f(x) \geq 0$. (given) So, $f(x) = 0, \quad \forall\ x \geq 1$

$\therefore\ [f(\sqrt{e}) + f(\sqrt{\pi})] = 0$ **Ans. :** (A)

(62) If $y(n) = e^x\, e^{x^2} ...\ e^{x^n}$, $0 < x < 1$, then the value of $\lim\limits_{n\to\infty} \dfrac{dy(n)}{dx}$ at $x = \dfrac{1}{2}$ = ...

(A) e (B) $4e$ (C) $2e$ (D) $3e$

Sol. : $y(n) = e^{x + x^2 + \cdots + x^n} = e^{\frac{x(1-x^n)}{1-x}}$

$\therefore\ \dfrac{dy(n)}{dx} = e^{\frac{x(1-x^n)}{1-x}}\ \dfrac{d}{dx}\left(\dfrac{x(1-x^n)}{1-x}\right)$

$= e^{\frac{x-x^{n+1}}{1-x}}\ \dfrac{d}{dx}\left(\dfrac{x}{1-x} - \dfrac{x^{n+1}}{1-x}\right)$

$$\lim_{n\to\infty} \frac{dy(n)}{dx} = \lim_{n\to\infty} e^{\left(\frac{x}{1-x}-\frac{x^{n+1}}{1-x}\right)} \frac{d}{dx}\left[\lim_{n\to\infty}\left(\frac{x}{1-x}-\frac{x^{n+1}}{1-x}\right)\right]$$

$= e^{\frac{x}{1-x}} \frac{d}{dx}\left(\frac{x}{1-x}\right)$ ($0 < x < 1$. So, if $n \to \infty$ then $x^{n+1} \to 0$)

$= e^{\frac{x}{1-x}} \frac{1}{(1-x)^2}$

$\therefore \left(\lim_{a\to\infty} \frac{dy(n)}{dx}\right)_{x=\frac{1}{2}} = 4e$ **Ans. :** (B)

(63) If $f''(x) = \frac{cos(\log x)}{x}$, $f'(1) = 0$ and $y = f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}$ = ...

(A) $\frac{sin(\log x)}{cos\, x}$ (B) $sin\left(\log\left(\frac{2x+3}{3-2x}\right)\right)$

(C) $\frac{12}{(3-2x)^2} sin\left(\log\left(\frac{2x+3}{3-2x}\right)\right)$ (D) $\frac{1}{(3-2x)^2} sin\left(\log\left(\frac{2x+3}{3-2x}\right)\right)$

Sol. : $f''(x) = \frac{cos(\log x)}{x} = \frac{d}{dx}(sin(\log x) + k)$

$\therefore f'(x) = sin(\log x) + k$. But $f'(1) = 0$. Hence, $k = 0$.

$\therefore f'(x) = sin(\log x)$

Now, $y = f\left(\frac{2x+3}{3-2x}\right)$

$\therefore \frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right) \frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$

$= f'\left(\frac{2x+3}{3-2x}\right)\left(\frac{2(3-2x)+2(2x+3)}{(3-2x)^2}\right)$

$= sin\left(\log\left(\frac{2x+3}{3-2x}\right)\right) \frac{12}{(3-2x)^2}$ **Ans. :** (C)

(64) If $f(x) = \begin{cases} ax^2 - b, & |x| < 1 \\ \frac{1}{|x|}, & |x| \geq 1 \end{cases}$ is differentiable at $x = 1$, then a = ..., b = ...

(A) $\frac{-1}{2}, \frac{-3}{2}$ (B) $\frac{-1}{2}, \frac{3}{2}$ (C) $\frac{1}{2}, \frac{-3}{2}$ (D) $\frac{1}{2}, \frac{3}{2}$

Sol. : $f'(1-) = (2ax)_{x=1} = 2a$

$f'(1+) = \left(\frac{-1}{x^2}\right)_{x=1} = -1$

But f is differentiable at $x = 1$

$\therefore\ 2a = -1.$ So, $a = -\frac{1}{2}$

$\therefore$ Moreover f is continuous at $x = 1$.

$\therefore\ \lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x)$

$\therefore\ \lim_{x\to 1-} (ax^2 - b) = \lim_{x\to 1+} \frac{1}{x}$

$\therefore\ a - b = 1$ So, $b = \frac{-3}{2}$ **Ans. :** (A)

(65) Let f be a function satisfying $f(x + y) = f(x) + f(y)$ and $f(x) = \sin x\ g(x)$, $\forall\ x, y \in R$. If $g(x)$ is continuous and $g(0) = c$, then $f'(x) =$ .....

(A) $c \sin n$ (B) c (C) $c \cos x$ (D) $\cos x\ g'(x)$

Sol. : $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{\sin h}{h}$ $g(h) = g(0) = c$ **Ans. :** (B)

(66) If $\lim_{x\to 0} \frac{e^{|x|+[x]} - 1}{a|x| + [x]}$ exists, then $a =$ .....

(A) 0 (B) $1 - e^{-1}$ (C) $(1 - e^{-1})^{-1}$ (D) $1 + e$

Sol. : If $x < 0$, let $-1 < x < 0$. So, $|x| = -x$, $[x] = -1$

$\lim_{x\to 0-} \frac{e^{|x|+[x]} - 1}{a|x| + [x]} = \lim_{x\to 0} \frac{e^{-x-1} - 1}{-ax - 1} = \frac{e^{-1} - 1}{-1} = 1 - \frac{1}{e}$

If $x > 0$, let $0 < x < 1$. So, $|x| = x$, $[x] = 0$

$\lim_{x\to 0+} \frac{e^{|x|+[x]} - 1}{a|x| + [x]} = \lim_{x\to 0+} \frac{e^x - 1}{ax} = \frac{1}{a}$

If limit exists, we should have, $1 - e^{-1} = \frac{1}{a}$

$\therefore\ a = (1 - e^{-1})^{-1}$ **Ans. :** (C)

(67) $\lim_{x\to 0-} \left[\frac{[x] + [x^2] + [x^3] + ... + [x^{2n+1}] + n + 1}{1 + [x] + |x| + 2x}\right] =$ .....

(A) 0 (B) -1 (C) 1 (D) limit does not exist

Sol. : $\lim_{x\to 0-} [x] = -1$, $[x^2] = 0$, $[x^3] = -1$, ...

$\therefore$ Required limit $= \lim_{x\to 0-} \frac{(n+1)(-1) + n(0) + n + 1}{1 - 1 - x + 2x} = 0$ **Ans. :** (A)

(68) If a quadratic equation $f(x) = ax^2 + bx + c = 0$ $(a \neq 0)$ has positive roots which are reciprocals of each other, then

(A) $f'(1) = 0$ (B) $af'(1) < 0$

(C) $af'(1) > 0$ (D) cannot say anything for $af'(1)$

Sol. : Consider $f(x) = ax^2 + bx + c = 0$ $(a \neq 0)$

Let α and $\frac{1}{\alpha}$ be two positive roots of $f(x)$.

$\therefore \alpha + \frac{1}{\alpha} = \frac{-b}{a} > 2$ $(\alpha > 0)$

$\therefore \frac{2a+b}{a} < 0$. So, $a(2a + b) < 0$

$\therefore a f'(1) = a(2a + b) < 0$ $(f'(x) = 2ax + b)$ **Ans. :** (B)

(69) $\lim\limits_{x\to\infty} \frac{\log_{10}(x+2)+[x+2]}{\left[x+\sum\limits_{r=0}^{\infty}\left(\frac{1}{2}\right)^r\right]}$ = ... where $[\cdot]$ is integer part function.

(A) 0 (B) 1 (C) -1 (D) limit does not exist

Sol. : $\sum\limits_{r=0}^{\infty}\left(\frac{1}{2}\right)^r = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots = \frac{1}{1-\frac{1}{2}} = 2$

$\therefore [x + 2] = [x] + 2$

$\therefore$ Required limit $= \lim\limits_{x\to\infty} \frac{\log_{10}(x+2)+[x]+2}{[x]+2}$

$= \lim\limits_{x\to\infty} \frac{\log_{10}(x+2)}{[x]+2} + 1 = 0 + 1 = 1$ **Ans. :** (B)

(70) If $f(x) = \sum\limits_{r=1}^{n} \tan^{-1}\left(\frac{1}{x^2+(2r-1)x+(r^2-r+1)}\right)$, then $\lim\limits_{n\to\infty} f'(0)$ = ... $(x > 0)$

(A) $\frac{3}{2}$ (B) $-\frac{e}{2}$ (C) e (D) -1

Sol. : $f(x) = \sum\limits_{r=1}^{n} \tan^{-1}\left(\frac{(x+r)-(x+r-1)}{1+(x+r)(x+r-1)}\right)$

$= \sum\limits_{r=1}^{n} [\tan^{-1}(x + r) - \tan^{-1}(x + r - 1)]$

$f(x) = \tan^{-1}(x + n) - \tan^{-1} x$. So, $f'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

$\therefore f'(0) = \frac{1}{1+n^2} - 1$

$\therefore \lim\limits_{n\to\infty} f'(0) = -1$ **Ans. :** (D)

(71) Let $f(x)$ and $g(x)$ be differentiable functions on R. If $f(2) = 8$, $g(2) = 0$, $f(4) = 10$ and $g(4) = 8$, then

(A) $g'(x) > 4f'(x)$, $\forall\ x \in (2, 4)$ (B) $g(x) > f(x)$, $\forall\ x \in (2, 4)$

(C) $3g'(x) = 4f'(x)$ for some $x \in (2, 4)$ (D) $g'(x) = 4f'(x)$ for some $x \in (2, 4)$

Sol. : Let $h(x) = g(x) - 4f(x)$

$h(2) = g(2) - 4f(2) = 0 - 4 \cdot 8 = -32$

$h(4) = g(4) - 4f(4) = 8 - 4 \cdot 10 = -32$

Moreover, h is continuous on $[2, 4]$ and differentiable on $(2, 4)$.

$\therefore$ The function h satisfies conditions of Rolle's theorem.

$\therefore$ For some $x \in (2, 4)$, $h'(x) = 0$

$\therefore$ $g'(x) = 4f'(x)$ for some $x \in (2, 4)$. **Ans. :** (D)

(72) Which of the following function does not satisfy conditions of mean value theorem on $[0, 1]$?

(A) $f(x) = \begin{cases} x, & x < \frac{1}{2} \\ \frac{1}{2}\left(\frac{1}{2} + x\right)^2, & x \geq \frac{1}{2} \end{cases}$ (B) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(C) $f(x) = (x^2 - 3x + 2)\,|x - 1|$ (D) $f(x) = |2x - 1|$

Sol. : From given functions in options by observing the functions, $f(x) = |2x - 1|$ is not differentiable at $x = \frac{1}{2}$.

$\therefore$ Function defined in (D) does not satisfy conditions of Mean Value theorem. **Ans. :** (D)

(73) $f : R \to R$ is a polynomial of degree n, where $f(x) = 0$ has n distinct real roots. How many real roots does the equation $[f'(x)]^2 - f(x)f''(x) = 0$ have ?

(A) No real root (B) One (C) Two (D) n

Sol. : Let $f(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$ and $h(x) = \frac{f'(x)}{f(x)}$, $x_i \in R$ for $i = 1, 2, 3, \ldots, n$

$\therefore$ $\log f(x) = \log(x - x_1) + \log(x - x_2) + \ldots\ldots + \log(x - x_n)$

$\therefore$ $h(x) = \frac{f'(x)}{f(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \ldots + \frac{1}{x - x_n}$

$$h'(x) = \frac{f(x)f''(x) - [f'(x)]^2}{[f(x)]^2} = -\left[\frac{1}{(x - x_1)^2} + \frac{1}{(x - x_2)^2} + \ldots + \frac{1}{(x - x_n)^2}\right] < 0,\ \forall\ x$$

$\therefore$ $f(x)f''(x) - [f'(x)]^2 < 0$, $\forall\ x$

$\therefore$ The given equation does not have any real root. **Ans. :** (A)

(74) $\lim\limits_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^4} = \ldots\ldots$

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{12}$

Sol. : Required limit $= \lim\limits_{x \to 0} \frac{2\sin\left(\frac{\sin x + x}{2}\right)\sin\left(\frac{x - \sin x}{2}\right)}{x^4}$

$$= 2 \lim_{x\to 0} \frac{\sin\left(\frac{\sin x + x}{2}\right)}{\frac{\sin x + x}{2}} \times \left(\frac{\sin x + x}{2}\right) \times \lim_{x\to 0} \frac{\sin\left(\frac{x - \sin x}{2}\right)}{\frac{x - \sin x}{2}} \times \left(\frac{x - \sin x}{2}\right) \times \frac{1}{x^4}$$

$$= 2 \lim_{x\to 0} \frac{x^2 - \sin^2 x}{4x^4}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{(x - \sin x)(x + \sin x)}{x^4}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{x - \sin x}{x^3} \times \lim_{x\to 0} \left(1 + \frac{\sin x}{x}\right)$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{x - \sin x}{x^3} \times 2 \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$$ **Ans. :** (C)

(75) If $x = a \cos t \sqrt{\cos 2t}$ and $y = a \sin t \sqrt{\cos 2t}$ $(a > 0)$, then $\left|\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right|_{t=\frac{\pi}{6}}$ = ...

(A) $\frac{a}{3}$ (B) $a\sqrt{2}$ (C) $\frac{\sqrt{2}}{3a}$ (D) $\frac{\sqrt{2}a}{3}$

Sol. : $x = a \cos t \sqrt{\cos 2t}$

$$\frac{dx}{dt} = a\left[\cos t \frac{1}{2\sqrt{\cos 2t}}(-2\sin 2t) + \sqrt{\cos 2t}\,(-\sin t)\right]$$

$$= \frac{-a}{\sqrt{\cos 2t}} [\sin 2t \cos t + \sin t \cos 2t] = \frac{-a \sin 3t}{\sqrt{\cos 2t}}$$

Similarly $\frac{dy}{dt} = \frac{a\cos 3t}{\sqrt{\cos 2t}}$. So, $\frac{dy}{dx} = -\cot 3t$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 3t \cdot 3 \frac{\sqrt{\cos 2t}}{a \sin 3t} \qquad \left(\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}\right)$$

$$= -\frac{3}{a} \operatorname{cosec}^3 3t \sqrt{\cos 2t}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2 3t = \operatorname{cosec}^2 3t$$

Now, $\left|\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right| = \frac{a}{3}\left|\frac{cosec^2 3t}{cosec^3 3t\sqrt{cos 2t}}\right| = \frac{a}{3}\left|\frac{1}{cosec\, 3t\sqrt{cos 2t}}\right|$

Required value at $t = \frac{\pi}{6}$ is $\frac{a}{3}\left|\frac{1}{1\sqrt{\frac{1}{2}}}\right| = \frac{\sqrt{2}a}{3}$ **Ans. :** (D)

(76) $\frac{d^2x}{dy^2} =$ ..... (AIEEE : 2011)

(A) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (B) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (C) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (D) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

Sol. : $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\}$

$= \frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\}\frac{dx}{dy} = -\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1} = -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}$ **Ans. :** (B)

(77) If $f(x) = \prod_{r=1}^{18}(x-r)^{r^2(2016-r)}$, then $\frac{f'(2016)}{f(2016)}$ is ..... .

(A) a prime (B) an odd composite number (C) an irrational number (D) 0

Sol. : $\log\{f(x)\} = \sum_{r=1}^{18} r^2(2016-r)\log(x-r)$

$\frac{1}{f(x)}f'(x) = \sum_{r=1}^{18}\frac{r^2(2016-r)}{x-r}$

Putting $x = 2016$, we get $\frac{f'(2016)}{f(2016)} = \sum_{r=1}^{18}\frac{r^2(2016-r)}{2016-r} = \sum_{r=1}^{18} r^2 = \frac{18\cdot19\cdot37}{6} = 57 \times 37$

Ans. : (B)

(78) If $D(x) = \begin{vmatrix} tan\,x & tan(x+h) & tan(x+2h) \\ tan(x+2h) & tan\,x & tan(x+h) \\ tan(x+h) & tan(x+2h) & tan\,x \end{vmatrix}$, then $\lim_{h\to0}\frac{D\left(\frac{\pi}{3}\right)}{h^2\sqrt{3}} =$ .....

(A) 144 (B) 168 (C) 138 (D) 108

Sol. : Applying $C_3 - C_2$ and $C_2 - C_1$ and then $C_2\left(\frac{1}{h}\right)$, $C_3\left(\frac{1}{h}\right)$, we get

$$\frac{D(x)}{h^2} = \begin{vmatrix} tan\,x & \frac{tan(x+h)-tan\,x}{h} & \frac{tan(x+2h)-tan(x+h)}{h} \\ tan(x+2h) & \frac{tan\,x-tan(x+2h)}{h} & \frac{tan(x+h)-tanx}{h} \\ tan(x+h) & \frac{tan(x+2h)-tan(x+h)}{h} & \frac{tan\,x-tan(x+2h)}{h} \end{vmatrix}$$

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} = \sec^2 x$$

$$\lim_{h\to 0}\frac{\tan(x+2h)-\tan(x+h)}{h} = \sec^2 x$$

$$\lim_{h\to 0}\frac{-\tan(x+2h)+\tan x}{h} = -2\sec^2 x$$

$$\therefore \lim_{h\to 0}\frac{D(x)}{h^2} = \begin{vmatrix} \tan x & \sec^2 x & \sec^2 x \\ \tan x & -2\sec^2 x & \sec^2 x \\ \tan x & \sec^2 x & -2\sec^2 x \end{vmatrix}$$

$$= \begin{vmatrix} \tan x & \sec^2 x & \sec^2 x \\ 0 & -3\sec^2 x & 0 \\ 0 & 0 & -3\sec^2 x \end{vmatrix} \qquad (R_2 - R_1, R_3 - R_1)$$

$$= 9\tan x \sec^4 x$$

$$\therefore \lim_{h\to 0}\frac{D\left(\frac{\pi}{3}\right)}{h^2\sqrt{3}} = 144$$ **Ans. :** (A)

(79) If $a + b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has ..... in the interval (0, 1).

(A) at least one real root (B) one real and one complex root

(C) no root (D) two pure imaginary roots

Sol. : $f(x) = ax^3 + bx^2 + cx$ is continuous and differentiable in [0, 1].

$f'(x) = 3ax^2 + 2bx + c$

Morevoer, $f(0) = 0$, $f(1) = a + b + c = 0$. So, $f(0) = f(1)$

By Rolle's theorem, there exists $k \in (0, 1)$ such that $f'(k) = 0$.

$\therefore 3ak^2 + 2bk + c = 0$ for $x = k$.

$\therefore k$ is a root of $3ax^2 + 2bx + c = 0$ lying in (0, 1). **Ans. :** (A)

(80) Between two roots of $e^{-x} - \sin x = 0$, the equation $e^x \cos x = -1$ has

(A) at least one real root (B) a root in $0 < x < \frac{3\pi}{2}$

(C) a root in $\frac{3\pi}{2} < x < 2\pi$ (D) no root

Sol. : Let $f(x) = e^{-x} - \sin x$ and α, β ($\alpha < \beta$) be two roots of $f(x) = 0$. $f(x)$ is continuous and differentiable in $[\alpha, \beta]$. Also $f(\alpha) = f(\beta) = 0$

By Rolle's theorem there exists at least one $k \in (\alpha, \beta)$ such that $f'(k) = 0$.

$\therefore f'(x) = -e^{-x} - \cos x$

$f'(k) = 0 \Rightarrow -e^{-k} - \cos k = 0 \Rightarrow -e^{-k}(1 + e^k \cos k) = 0 \Rightarrow e^k \cos k = -1$

$\therefore k$ is a root of $e^x \cos x = -1$. **Ans. :** (A)

(81) $\lim_{x\to 0}(1 - x + [x - 1] + [1 - x]) =$ .....

where $[\cdot]$ is integer part function.

(A) 0 (B) 1 (C) –1 (D) limit does not exist

Sol. : Let $f(x) = 1 - x + [x - 1] + [1 - x]$

$\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0} (1 - x - 1 + 0) = 0$ (Taking $0 < x < 1$)

$\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} (1 - x - 2 + 1) = 0$ ($-1 < x < 0$)

$\therefore \lim\limits_{x\to 0} f(x) = 0$ **Ans. :** (A)

(82) If $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$, then among roots of $f'(x) = 0$ ..... .

(A) all the three roots are positive (B) all the three roots are negative

(C) two non-real complex numbers (D) one root negative, two roots positive

Sol. : $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$

Applying Rolle's theorem on [1, 2], [2, 3], [3, 4]. $f'(x) = 0$ will have a root in (1, 2), (2, 3), (3, 4)

Ans. : (A)

(83) If $f(x) = \begin{cases} \int_0^x (1+|1-t|)\,dt, & x > 2 \\ 5x-7, & x \le 2 \end{cases}$, then

(A) f is not continuous at $x = 2$ (B) f is differentiable on R

(C) $\lim\limits_{x\to 2+} f(x)$ does not exist (D) f is continuous but not differentiable at $x = 2$

Sol. : For $x > 2$, $f(x) = \int_0^1 (1+(1-t))\,dt + \int_1^x (1+(t-1))\,dt$

$= \int_0^1 (2-t)\,dt + \int_1^x t\,dt = \left[2t - \frac{t^2}{2}\right]_0^1 + \left[\frac{t^2}{2}\right]_1^x = \frac{3}{2} + \frac{x^2}{2} - \frac{1}{2}$

$= \frac{x^2}{2} + 1$

$\therefore f(x) = \begin{cases} \frac{x^2}{2}+1, & x > 2 \\ 5x-7, & x \le 2 \end{cases}$

$\lim\limits_{x\to 2+} f(x) = \lim\limits_{x\to 2+} \left(\frac{x^2}{2}+1\right) = 3,$

$\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} (5x - 7) = 3.$ Moreover, $f(2) = 3$

$\therefore \lim\limits_{x\to 2} f(x) = f(2)$ So, f is continuous at $x = 2$

$f'(x) = \begin{cases} x, & x > 2 \\ 5, & x < 2 \end{cases}$ So, $f'(2+) = 2$, $f'(2-) = 5$

$\therefore f'(2-) \neq f'(2+)$

$\therefore f$ is not differentiable at $x = 2$ **Ans. :** (D)

(84) $f : [0, 4] \to R$ is a differentiable function. If there exist $a, b \in (0, 4)$ such that $(f(4))^2 - (f(0))^2 = k f'(a) f(b)$, then k = ...

(A) 8 (B) not possible (C) 4 (D) 1

Sol. : By using mean value theorem there exist $a \in (0, 4)$ such exist $\frac{f(4)-f(0)}{4-0} = f'(a)$

$\therefore f(4) - f(0) = 4f'(a)$ (1)

Moreover, function f is continuous on $[0, 4]$. So, it attains all values between its minimum and maximum values.

Now, $\frac{f(4)+f(0)}{2} \in (f(0), f(4))$

$\therefore$ There exists some $b \in (0, 4)$ such that $\frac{f(4)+f(0)}{2} = f(b)$

$\therefore f(4) + f(0) = 2 f(b)$ (2)

From (1) and (2), $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$

$\therefore k = 8$ **Ans. :** (A)

(85) $\lim\limits_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ = ...

(A) 2 (B) –2 (C) $\frac{1}{2}$ (D) –1

Sol. : Required limit = $\lim\limits_{x\to 0} \frac{x}{4\sin^4 x}\left[\frac{2\tan x}{1-\tan^2 x} - 2\tan x\right]$

$= \lim\limits_{x\to 0} \frac{2\tan x}{4x^3}\left[\frac{1}{1-\tan^2 x} - 1\right]$ $\left(\lim\limits_{x\to 0}\frac{\sin x}{x} = 1\right)$

$= \lim\limits_{x\to 0} \frac{1}{2x^2}\left(\frac{1-1+\tan^2 x}{1-\tan^2 x}\right)$ $\left(\lim\limits_{x\to 0}\frac{\tan x}{x} = 1\right)$

$= \frac{1}{2}\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^2 \times \lim\limits_{x\to 0}\frac{1}{1-\tan^2 x} = \frac{1}{2}$ **Ans. :** (C)

(86) Let a function f be differentiable on R. If $f(1) = -2$ and $f'(x) \geq 2, \forall x \in [1, 6]$,

(A) $f(6) < 2$ (B) $f(6) \geq 8$ (C) $f(6) = 5$ (D) $f(6) < 5$

Sol. : By using Mean Value theorem,

$\frac{f(6)-f(1)}{6-1} = f'(c)$ where $c \in (1, 6)$

$\frac{f(6)-(-2)}{5} \geq 2$ $(f'(x) \geq 2, \ \forall x \in [1, 6])$

$\therefore f(6) \geq 8$ **Ans. :** (B)

(87) If $\lim\limits_{n\to\infty} \frac{(1^a + 2^a + ... + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+...+(na+n)]} = \frac{1}{60}$, then a = ... $(a \neq -1)$ (JEE Adv. : 2013)

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

Sol. : Given limit $= \lim_{n\to\infty} \dfrac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\left\{n^2 a+\dfrac{n(n+1)}{2}\right\}}$

$= \lim_{n\to\infty} \dfrac{\sum_{r=1}^{n} r^a}{n^{a-1}\cdot n^2\left(1+\dfrac{1}{n}\right)^{a-1}\left\{a+\dfrac{1}{2}+\dfrac{1}{2n}\right\}}$

$= \lim_{n\to\infty} \dfrac{\dfrac{1}{n}\sum_{r=1}^{n}\left(\dfrac{r}{n}\right)^a}{\left(1+\dfrac{1}{n}\right)^{a-1}\left\{a+\dfrac{1}{2}+\dfrac{1}{2n}\right\}}$

$\therefore \quad \dfrac{1}{60} = \dfrac{\int_0^1 x^a\, dx}{a+\dfrac{1}{2}} = \dfrac{1}{(a+1)\left(a+\dfrac{1}{2}\right)}$

$\therefore$ $(2a + 1)(a + 1) = 120$

$\therefore$ $2a^2 + 3a - 119 = 0$. So, $(2a + 17)(a - 7) = 0$. Hence, $a = 7$ or $a = \dfrac{-17}{2}$

But if $a > -1$, then only $\int_0^1 x^a\, dx$ exists.

$\therefore$ $a = 7$ **Ans. :** (B)

(88) Which of the following functions does not satisfy conditions of Mean Value theorem in [0, 1] ?

(A) $f(x) = \begin{cases} \dfrac{1}{2}-x, & x<\dfrac{1}{2} \\ \left(\dfrac{1}{2}-x\right)^2, & x\ge\dfrac{1}{2} \end{cases}$ (B) $f(x) = |x|$ (C) $f(x) = x\,|x|$ (D) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x\neq 0 \\ 1, & x=0 \end{cases}$

Sol. : The function defined in option (A) is not differentiable at $x = \dfrac{1}{2}$. Therefore it does not satisfy conditions of Mean Value theorem in [0, 1]. **Ans. :** (A)

(89) If $f''(0) = 4$, then $\lim_{x\to 0} \dfrac{2f(x)-3f(2x)+f(4x)}{x^2}$ = ...

(A) 6 (B) –6 (C) 12 (D) –12

Sol. : $\lim_{x\to 0} \dfrac{2f(x)-3f(2x)+f(4x)}{x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0} \dfrac{2f'(x)-6f'(2x)+4f'(4x)}{2x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0} \dfrac{2f''(x)-12f''(2x)+16f''(4x)}{2} = 3f''(0) = 12$ **Ans. :** (C)

(90) If $f(x) = \begin{cases} \dfrac{a|x^2-x-2|}{2+x-x^2}, & x<2 \\ b, & x=2 \\ \dfrac{x-[x]}{x-2}, & x>2 \end{cases}$ is continuous at $x = 2$, then (a, b) = ...

(A) (1, 2) (B) (1, 1) (C) (2, 1) (D) (2, 2)

Sol. : $x^2 - x - 2 = (x + 1)(x - 2)\begin{cases} <0 & x\in(-1,2) \\ >0, & x>2 \end{cases}$

$\therefore$ If $-1 < x < 2$, then $|x^2 - x - 2| = 2 + x - x^2$

Moreover, if $x > 2$, then $[x] = 2$ $(2 < x < 3)$

$\therefore$ We can take $f(x) = \begin{cases} a, & -1<x<2 \\ b, & x=2 \\ 1, & 2<x<3 \end{cases}$

$\therefore$ If $f(x)$ is continuous at $x = 2$, then $(a, b) = (1, 1)$ **Ans. :** (B)

(91) If $\lim\limits_{x\to0} \dfrac{\sin(\sin x)-\sin x}{ax^3+bx^5+c} = \dfrac{-1}{12}$, then

(A) $a = 2, b \in R, c = 0$ (B) $a = -2, b \in R, c = 0$

(C) $a = 1, b \in R, c = 0$ (D) $a = -1, b \in R, c = 0$

Sol. : Given limit $= \lim\limits_{x\to0} \dfrac{2\cos\left(\dfrac{\sin x+x}{2}\right)\sin\left(\dfrac{\sin x-x}{2}\right)}{ax^3+bx^5+c}$

$\therefore \dfrac{-1}{12} = \lim\limits_{x\to0} \dfrac{\sin x-x}{ax^3+bx^5+c}$ $\left(\lim\limits_{x\to0}\cos\left(\dfrac{\sin x+x}{2}\right)=1,\ \lim\limits_{x\to0}\dfrac{\sin\left(\dfrac{\sin x-x}{2}\right)}{\left(\dfrac{\sin x-x}{2}\right)}=1\right)$

$\therefore \dfrac{-1}{12} = \lim\limits_{x\to0} \dfrac{\dfrac{-x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{-x^7}{7!}+\ldots}{ax^3+bx^5+c} = \lim\limits_{x\to0} \dfrac{\dfrac{-1}{6}+\dfrac{x^2}{120}+\ldots..+\infty}{a+bx^2+\dfrac{c}{x^3}}$

$\therefore c = 0, b \in R, \dfrac{-1}{6a} = \dfrac{-1}{12}$, So, $a = 2$ **Ans. :** (A)

(92) If $f(x) = \lim\limits_{n\to\infty} \dfrac{\log_e(2+x)-x^{2n}\sin x}{1+x^{2n}}$, then

(A) $f(x)$ is continuous at $x = 1$. (B) $\lim\limits_{x\to1+} f(x) = \log_e 3$.

(C) $\lim\limits_{x\to1+} f(x) = -\sin 1$. (D) $\lim\limits_{x\to1-} f(x)$ does not exist

Sol. : If $|x| < 1$, then $\lim\limits_{n\to\infty} x^{2n} = 0$. If $|x| > 1$, then $\lim\limits_{n\to\infty} x^{-2n} = 0$

$\therefore\ f(x) = \begin{cases} \log_e(2+x), & |x|<1 \\ -\sin x\ , & |x|>1 \end{cases}$

$\lim\limits_{x\to 1+} f(x) = -\sin 1,\ \lim\limits_{x\to 1-} f(x) = \log 3$ **Ans. :** (C)

(93) $\lim\limits_{x\to 1} \left\{ e^{\left(\frac{4}{\pi}\tan^{-1}x - 1\right)} \right\}^{\frac{1}{x^2-1}} = ...$

(A) e^{π} (B) $e^{\frac{2}{\pi}}$ (C) $e^{\frac{1}{\pi}}$ (D) $e^{\frac{-1}{\pi}}$

Sol. : Required limit $= \lim\limits_{x\to 1} \left\{ e^{\left(\frac{4}{\pi}\tan^{-1}x - 1\right)} \right\}^{\frac{1}{x^2-1}}$ $\left(\frac{0}{0}\text{ form}\right)$

$= e^{\lim\limits_{x\to 1}\left(\frac{\frac{4}{\pi}\frac{1}{1+x^2}}{2x}\right)} = e^{\frac{1}{\pi}}$ **Ans. :** (C)

(94) If $f(x) = \begin{cases} a+\sin^{-1}(x+b), & x>1 \\ x, & x\le 1 \end{cases}$ is differentiable at $x = 1$, then a = ..., b = ...

(A) 1, 1 (B) –1, –1 (C) 1, –1 (D) –1, 1

Sol. : $f'(x) = \begin{cases} \frac{1}{\sqrt{1-(x+b)^2}}, & x>1 \\ 1, & x<1 \end{cases}$

$f'(1+) = f'(1-)$ (f is differentiable at $x = 1$)

So, $\frac{1}{\sqrt{1-(1+b)^2}} = 1$

$\therefore\ 1 = \sqrt{1-(1+b)^2}$

$\therefore\ 1 = 1 - (1+b)^2$

$\therefore\ (1+b)^2 = 0.$ Hence, $b = -1$

Moreover, f is continuous at $x = 1$ also.

$\therefore\ \lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1-} f(x) = f(1)$

$\therefore\ a + \sin^{-1}(1-1) = 1$ ($b = -1$)

$\therefore\ a = 1.$ So, $a = 1$, $b = -1$ **Ans. :** (C)

(95) If $f(x) = \begin{cases} 1+x, & 0\le x\le 2 \\ 3-x, & 2<x\le 3 \end{cases}$ then at how many points is fof not differentiable ?

(A) 2 (B) 1 (C) 3 (D) 0

Sol. : The function f is not differentiable at $x = 2$ as f is not continuous at $x = 2$.

Now, $(fof)(x) = f(f(x)) = \begin{cases} f(1+x), & 0\le x\le 2 \\ f(3-x), & 2<x\le 3 \end{cases}$

If $0 \le x \le 1$, then $1 \le 1 + x \le 2$

$\therefore\ f(f(x)) = f(1 + x) = 1 + 1 + x = 2 + x$

If $1 \le x \le 2$, then $2 \le 1 + x \le 3$

$\therefore\ f(f(x)) = f(1 + x) = 3 - (1 + x) = 2 - x$

$2 < x \le 3 \Rightarrow -3 < -x \le -2 \Rightarrow 0 < 3 - x \le 1$

$\therefore\ f(f(x)) = f(3 - x) = 1 + 3 - x = 4 - x$

$\therefore$ (*fof*) is not differentiable at $x = 1$ as well as $x = 2$.

fof is not differentiable at two points. **Ans. :** (A)

(96) The function $g(x)$ is the inverse of function $f(x)$. If the function $f(x)$ is differentiable on R, then $g''(f(x))$ = ...

(A) $\frac{-f''(x)}{(f'(x))^3}$ (B) $\frac{f'(x)f''(x)-(f'(x))^2}{f'(x)}$ (C) $\frac{f'(x)f''(x)-(f'(x))^2}{(f'(x))^2}$ (D) $\frac{f''(x)}{(f'(x))^3}$

Sol. : We have $g(x) = f^{-1}(x)$ So, $f(g(x)) = g(f(x)) = x$

$\therefore\ x = g(f(x))$

Differentiating,

$1 = g'(f(x))\, f'(x)$

$\therefore\ g'(f(x)) = \frac{1}{f'(x)}$

Differentiating,

$$g''(f(x))\, f'(x) = -\frac{1}{(f'(x))^2}\, f''(x)$$

$$\therefore\ g''(f(x)) = -\frac{f''(x)}{(f'(x))^3}$$ **Ans. :** (A)

(97) If $\lim\limits_{x\to\infty} \left(\sqrt{x^6+ax^5+bx^3+cx+d} - \sqrt{x^6-2x^5+x^3-x+1}\right) = 2$, then

(A) $b = -2$ (B) $a = -2$ (C) $a = 2$ (D) $b = -5$

Sol. : Given limit $= \lim\limits_{x\to\infty} \frac{(a+2)x^5+(b-1)x^3+(c+1)x+d-1}{\sqrt{x^6+ax^5+bx^3+cx+d}+\sqrt{x^6-2x^5+x^3-x+1}}$

$$\therefore\ 2 = \lim_{x\to\infty} \frac{(a+2)x^2+(b-1)+\left(\frac{c+1}{x^2}\right)+\left(\frac{d-1}{x^3}\right)}{\left(\sqrt{1+\frac{a}{x}+\frac{b}{x^3}+\frac{c}{x^5}+\frac{d}{x^6}}+\sqrt{1-\frac{2}{x}+\frac{1}{x^3}-\frac{1}{x^5}+\frac{1}{x^6}}\right)}$$

Since limit exists, we have, $a + 2 = 0$, $\frac{b-1}{2} = 2$, $c, d \in \mathrm{R}$

$\therefore\ a = -2, b = 5,\ c, d \in \mathrm{R}$ **Ans. :** (B)

(98) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\log_e(\sin x)} = ...$

(A) 1 (B) 0 (C) 2 (D) –1

Sol. : Let $\sin x = t$. If $x \to \frac{\pi}{2}$, then $t \to 1$

Required limit $= \lim\limits_{t\to1} \dfrac{t-t^t}{1-t+\log_e t}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{t\to1} \dfrac{1-t^t(1+\log_e t)}{-1+\frac{1}{t}}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{t\to1} \dfrac{-t^t(1+\log_e t)^2-t^t\cdot\frac{1}{t}}{-\frac{1}{t^2}} = \dfrac{-1-1}{-1} = 2$ **Ans. :** (C)

(99) $\lim\limits_{x\to\infty} \dfrac{\sum\limits_{r=1}^{2016}(r+x)^{2016}}{x^{2016}+2016^{2016}} = ...$

(A) 2014 (B) 2015 (C) 2016 (D) limit does not exist

Sol. : $\lim\limits_{x\to\infty} \dfrac{\left\{(1+x)^{2016}+(2+x)^{2016}+...+(2016+x)^{2016}\right\}}{x^{2016}+2016^{2016}}$

$= \lim\limits_{x\to\infty} \dfrac{x^{2016}\left\{\left(\frac{1}{x}+1\right)^{2016}+\left(\frac{2}{x}+1\right)^{2016}+...+\left(\frac{2016}{x}+1\right)^{2016}\right\}}{x^{2016}\left(1+\left(\frac{2016}{x}\right)^{2016}\right)}$

$= \dfrac{(0+1)^{2016}+(0+1)^{2016}+...+(0+1)^{2016}}{1+0} = 2016$ $\left(\lim\limits_{x\to\infty}\frac{1}{x}=0\right)$ **Ans. :** (C)

(100) If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = x f(x)$, then $\left(\frac{dy}{dx}\right)_{x=1}$ = ...

(A) 14 (B) –1 (C) 1 (D) $\frac{7}{8}$

Sol. : $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ (1)

$5f'(x) + 3f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 1$ (2)

Now, $y = x f(x)$ so, $\frac{dy}{dx} = f(x) + xf'(x)$

$\left(\frac{dy}{dx}\right)_{x=1} = f(1) + f'(1)$

Substituting $x = 1$ in equation (1), we get, $5\ f(1) + 3f(1) = 3$. So, $f(1) = \frac{3}{8}$

Substituting $x = 1$ in equation (2), we get, $5f'(1) - 3f'(1) = 1$. So, $f'(1) = \frac{1}{2}$

$\therefore \left(\frac{dy}{dx}\right)_{x=1} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$ **Ans. :** (D)

(101) Let a function f be twice differentiable, where $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = (f(x))^2 + (g(x))^2$ and $h(5) = 3$, then $h(10) =$ .....

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $g(x) = f'(x)$

$\therefore g'(x) = f''(x) = -f(x)$

$h(x) = (f(x))^2 + (g(x))^2$

$\therefore h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)f'(x) + 2f'(x)(-f(x)) = 0$

$\therefore h(x)$ is a constant function.

Since $h(5) = 3$, we have, $h(10) = 3$ **Ans. :** (D)

(102) $\lim\limits_{x\to 0} \frac{\tan\sqrt[3]{x}\log(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2 (e^{5\sqrt[3]{x}} - 1)} =$ .....

(A) $\frac{\sqrt{3}}{5}$ (B) 1 (C) $\frac{3}{5}$ (D) 0

Sol. : Required limit $= \frac{\lim\limits_{x\to 0} \frac{\tan\sqrt[3]{x}}{\sqrt[3]{x}}\sqrt[3]{x} \times \lim\limits_{x\to 0} \frac{\log(1+3x)}{3x}\cdot 3x}{\lim\limits_{x\to 0}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\right)^2 \cdot \left(\sqrt{x}\right)^2 \times \lim\limits_{x\to 0} \frac{e^{5\sqrt[3]{x}} - 1}{5\sqrt[3]{x}} \times 5\sqrt[3]{x}} = \lim\limits_{x\to 0} \frac{\sqrt[3]{x}\cdot 3x}{x\ 5\sqrt[3]{x}} = \frac{3}{5}$ **Ans. :** (C)

(103) If $xy = e - e^y$, then $\left(\frac{d^2y}{dx^2}\right)_{x=0} =$ .....

(A) $\frac{1}{e}$ (B) $\frac{1}{e^3}$ (C) $\frac{1}{e^2}$ (D) e^2

Sol. : $xy = e - e^y$ (1)

$x\frac{dy}{dx} + y = -e^y\frac{dy}{dx}$ (2)

$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = -e^y\left(\frac{dy}{dx}\right)^2 - e^y\frac{d^2y}{dx^2}$ (3)

Substituting $x = 0$ in (1), we get $0 = e - e^{y(0)}$

$\therefore e^{y(0)} = e^1$. So, $y(0) = 1$

Substituting $x = 0$ in (2), we get, $y(0) = -e^{y(0)}\left(\frac{dy}{dx}\right)_{x=0}$

$1 = -e\left(\frac{dy}{dx}\right)_{x=0}$. So, $\left(\frac{dy}{dx}\right)_{x=0} = -\frac{1}{e}$

Substituting $x = 0$ in (3), we get, $2\left(\frac{dy}{dx}\right)_{x=0} = -e^{y(0)}\left[\left(\frac{dy}{dx}\right)_{x=0}\right]^2 - e^{y(0)}\left(\frac{d^2y}{dx^2}\right)_{x=0}$

$-\frac{2}{e} = -e\ \frac{1}{e^2} - e\left(\frac{d^2y}{dx^2}\right)_{x=0}$

$\therefore\ \frac{1}{e^2} = \left(\frac{d^2y}{dx^2}\right)_{x=0}$ **Ans. :** (C)

(104) Let $f(x) = \begin{cases} x^n \sin\frac{1}{x}, & x \neq 0 \\ 0\ , & x = 0 \end{cases}$ If $f(x)$ is continuous but not differentiable at $x = 0$, then

(A) $n \in (0, 1]$ (B) $n \in [1, \infty)$ (C) $n \in (-\infty, 0)$ (D) $n = 0$

Sol. : Since f is continuous at $x = 0$, we have,

$\lim\limits_{x\to 0} f(x) = f(0) = 0$

$\therefore\ \lim\limits_{x\to 0} x^n \sin\frac{1}{x} = 0$

$\therefore\ n > 0$ (If $n > 0$ then we can take $\delta = \varepsilon^{\frac{1}{n}}$)

If f is differentiable at $x = 0$, then

$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists.

$\therefore\ \lim\limits_{x\to 0} \frac{x^n \sin\frac{1}{x}-0}{x}$ exists.

$\therefore\ \lim\limits_{x\to 0} x^{n-1} \sin\frac{1}{x}$ exists.

$\therefore\ n - 1 > 0$. So, $n > 1$

If $n \leq 1$, then $\lim\limits_{x\to 0} x^{n-1} \sin\frac{1}{x}$ does not exist and the therefore f is not differentiable at $x = 0$.

$\therefore$ If $n \in (0, 1]$, then f is continuous but not differentiable at $x = 0$. **Ans. :** (A)

(105) If $f(x) = \left(\sum\limits_{k=0}^{n} a_k |x-1|^k\right)$ where $a_k > 0$, then $f(x)$

(A) not continuous at $x = 1$ (B) continuous on R but not differentiable at $x = 0$

(C) differentiable on R (D) continuous on R but not differentiable at $x = 1$

Sol. : $|x-1|$, $|x-1|^2$, ... are continuous on R. Since sum of two or more continuous functions is also continuous, $f(x)$ is continuous on R. Moreover, $|x-1|$, $|x-1|^3$, ... are not differentiable at $x = 1$.

$\therefore f(x)$ is continuous on R but not differentiable at $x = 1$. **Ans. :** (D)

(106) Let a real function f satisfy $f(x+y) = f(x) - f(y) + 2xy - 1$, $\forall\ x, y \in R$.

If f is differentiable and $f'(0) = b$, then for which value of b is the value of $f(x) > 0$, $\forall x$?

(A) No such b exists (B) $b > 0$ (C) $b < 0$ (D) $b = 0$

Sol. : Substituting $x = 0 = y$ in the functional equation $f(x+y) = f(x) - f(y) + 2xy - 1$, we get $f(0) = -1$.

$$\text{Now, } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)-f(h)+2xh-1-f(x)}{h}$$

$$= 2x - \lim_{x\to 0} \frac{f(h)-f(0)}{h} = 2x - f'(0) = 2x - b \qquad (f(0) = -1)$$

$$\therefore\ f'(x) = 2x - b$$

$\therefore f(x) = x^2 - bx + c$, where c is constant

Substituting $x = 0$, we get $c = -1$. So, $f(x) = x^2 - bx - 1$

$\forall\ x \in R$, if a $D = b^2 - 4ac = b^2 + 4 < 0$, then $f(x) > 0$. $(a = 1)$

which is not possible for any real value of b. **Ans. :** (A)

(107) If $y = \sin x \cdot \sin 2x \cdot \sin 3x \ldots \cdot \sin nx$, then $\frac{dy}{dx} = \ldots\ldots$

(A) $\sum_{k=1}^{n} k \tan kx$ (B) $y \sum_{k=1}^{n} k \cot kx$ (C) $y \sum_{k=1}^{n} k \tan kx$ (D) $\sum_{k=1}^{n} k \cot kx$

Sol. : $y = \sin x \cdot \sin 2x \cdot \sin 3x \ldots \cdot \sin nx$

$\therefore \log y = \log \sin x + \log \sin 2x + \ldots + \log \sin nx$

$$\frac{1}{y}\frac{dy}{dx} = \cot x + 2 \cot 2x + \ldots + n \cot nx$$

$$\therefore \frac{dy}{dx} = y \sum_{k=1}^{n} k \cot kx$$ **Ans. :** (B)

(108) If $f(x + y + z) = f(x) f(y) f(z)$, $\forall\ x, y, z \in R$, $f(4) = 4$, $f'(0) = 2$ then $f'(4)$, $f(4)$, $f'(0)$ are in .....

(A) arithmetic progression (B) geometric progression

(C) harmonic progression (D) (A), (B), (C) are true

Sol. : $f(x + y + z) = f(x) f(y) f(z)$ (1)

Substituting $x = 4$, $y = 0$, $z = 0$ in equation (1) we get,

$f(4) = f(4) f(0) f(0)$

$\therefore f(0) = \pm 1$ $(f(4) = 4 \neq 0)$

Again, substituting $x = 4$, $y = -2$, $z = -2$, in equation (1) we get,

$f(0) = f(4)\ [f(-2)]^2 = 4\ \left[f(-2)\right]^2 \geq 0$.

$\therefore f(0) \neq -1$ So, $f(0) = 1$

Substituting $y = 4$, $z = 0$ in equation (1), we get,

$f(x + 4) = f(x) f(4) f(0) = 4 f(x)$

$\therefore\ f'(x+4) = 4f'(x)$. So, $f'(4) = 4f'(0) = 8$. So, $f'(4) = 8, f(4) = 4, f'(0) = 2$

$\therefore\ f'(4), f(4), f'(0)$ are in geometric progression. **Ans. :** (B)

(109) If $\sqrt{r} = ae^{\theta \cot \alpha}$, then $\frac{d^2r}{d\theta^2} - 4r\cot^2\alpha = ...$ $(a, \alpha \in R)$

(A) r (B) $\frac{1}{r}$ (C) 1 (D) 0

Sol. : $\sqrt{r} = ae^{\theta \cot \alpha}$

$\therefore\ r = a^2 e^{2\theta \cot \alpha}$ (1)

$\therefore\ \frac{dr}{d\theta} = a^2 \cot\alpha \cdot 2e^{2\theta \cot \alpha}$

$\therefore\ \frac{d^2r}{d\theta^2} = 4a^2 \cot^2\alpha \cdot e^{2\theta \cot \alpha}$. So, $\frac{d^2r}{d\theta^2} - 4r\cot^2\alpha = 0$ **Ans. :** (D)

(110) If $f(x) = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$, then $f'(x) = ...$

(A) $\cos 2x$ (B) $-\cos 2x$ (C) $\sin 2x$ (D) $-\sin 2x$

Sol. : $f(x) = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$

$= \frac{\sin^2 x.\tan x}{1+\tan x} + \frac{\cos^2 x}{1+\tan x}$

$= \frac{\frac{\sin^3 x}{\cos x} + \cos^2 x}{1+\tan x}$

$= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = \sin^2 x - \sin x \cos x + \cos^2 x$

$f(x) = 1 - \frac{1}{2}\sin 2x$. So, $f'(x) = -\cos 2x$ **Ans. :** (B)

(111) If $f(x) = 1 - \cos x$, then $\lim_{x\to\infty} \left\{ e^{\sqrt{f\left(\frac{1}{x}\right)\sqrt{f\left(\frac{1}{x}\right)\sqrt{f\left(\frac{1}{x}\right)}\ldots}}} \right\}^{\frac{1}{\cos^{-1}\left\{1-f\left(\frac{1}{x^2}\right)\right\}}} = ...$

(A) e (B) 1 (C) $\sqrt{e}$ (D) $\sqrt{\sqrt{e}}$

Sol. : $f\left(\frac{1}{x^2}\right) = 1 - \cos\frac{1}{x^2}$ So, $\cos\frac{1}{x^2} = 1 - f\left(\frac{1}{x^2}\right)$

$\therefore \frac{1}{x^2} = cos^{-1}\left\{1-f\left(\frac{1}{x^2}\right)\right\}$ So, $x^2 = \frac{1}{cos^{-1}\left\{1-f\left(\frac{1}{x^2}\right)\right\}}$

Required limit $= \lim_{x\to\infty} \left\{ e^{\sqrt{(1-\cos\frac{1}{x})\sqrt{(1-\cos\frac{1}{x})\sqrt{(1-\cos\frac{1}{x})...}}}} \right\}^{x^2}$

$= \lim_{y\to 0} \left\{ e^{\sqrt{(1-\cos y)\sqrt{(1-\cos y)\sqrt{(1-\cos y)...}}}} \right\}^{\frac{1}{y^2}}$ $\left(\frac{1}{x}=y\right)$

$= \lim_{e^{y\to 0}} \frac{\sqrt{(1-\cos y)\sqrt{(1-\cos y)\sqrt{(1-\cos y)...}}}}{y^2}$

$= \lim_{e^{y\to 0}} \frac{1-\cos y}{y^2}$ $\left(\sqrt{x\sqrt{x\sqrt{x...}}}=x\right)$

$= e^{\frac{1}{2}} = \sqrt{e}$ **Ans. :** (C)

(112) At how many points is $f(x) = \frac{|x|}{|x|+2016}$ not differentiable ?

(A) 0 (B) 1 (C) 2 (D) infinitely many

Sol. : If $x < 0$, then $f(x) = \frac{-x}{-x+2016} = \frac{x}{x-2016}$

which is differentiable for all $x < 0$.

If $x > 0$, then $f(x) = \frac{x}{x+2016}$, which is differentiable for all $x > 0$.

Let us check differentiability at $x = 0$.

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{\frac{x}{x+2016}}{x} = \frac{1}{2016}$$

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{\frac{-x}{-x+2016}}{x} = -\frac{1}{2016}$$

$\therefore f$ is not differentiable at $x = 0$. **Ans. :** (B)

(113) If $y = {}_{2^{x^{2^x}}}$, then $\frac{dy}{dx}$ = ...

(A) $2^{x^{2^x}}\ x^{2^x} \log_e 2 \left(2^x \log_e x \log_e 2 + \frac{2^x}{x}\right)$ (B) $2^{x^{2^x}}\ (1 + x \log_e 2)\ 2^x \log_e 2$

(C) $2^{x^{2^x}}\ (1 + x \log_e 2)\ 2^x\ (\log_e 2)^2$ (D) $2^{x^{2^x}}\ (1 - x \log_e 2)\ 2^x \log_e 2$

Sol. : $y = 2^{x^{2^x}}$

$\therefore \log_e y = x^{2^x} \log 2$

$\therefore \log_e (\log_e y) = 2^x \log_e x + \log_e (\log_e 2)$

$\therefore \frac{1}{\log_e y} \frac{1}{y} \frac{dy}{dx} = 2^x (\log_e 2)(\log_e x) + 2^x \frac{1}{x} + 0$

$\therefore \frac{dy}{dx} = y \log_e y \left(2^x (\log_e 2)(\log_e x) + \frac{2^x}{x} \right)$

$= 2^{x^{2^x}} \; x^{2^x} \log_e 2 \left(2^x (\log_e x)(\log_e 2) + \frac{2^x}{x} \right)$ **Ans. :** (A)

(114) $f''(x) + f(x) = 0, \forall x$ for function $f(x)$ and $g(x) = [f(x)]^2 + [f'(x)]^2$ and $g(3) = 8$, then $g(8) =$ .....

(A) 0 (B) 3 (C) 5 (D) 8

Sol. : $g(x) = [f(x)]^2 + [f'(x)]^2$

$g'(x) = 2 f(x) f'(x) + 2 f'(x) f''(x)$

$= 2 f'(x) (0) = 0, \forall x$ So, $g'(x) = 0, \forall x$

$\therefore g(x)$ is constant.

Since $g(3) = 8$, we have, $g(8) = 8$ **Ans. :** (D)

(115) If a function $g(x)$ is the inverse of $f(x)$ and $f'(x) = \frac{1}{1+x^3}$, then $g'(x) =$ .....

(A) $g(x)$ (B) $1 + g(x)$ (C) $1 + [g(x)]^3$ (D) $\frac{1}{1+[g(x)]^3}$

Sol. : As g is the inverse of f.

$\therefore g(f(x)) = x$

$\therefore g'(f(x)) f'(x) = 1$

$\therefore g'(f(x)) = \frac{1}{f'(x)} = 1 + x^3$

$\therefore g'[f(f^{-1}(x))] = 1 + [f^{-1}(x)]^3$

$\therefore g'(x) = 1 + [g(x)]^3$

Second method :

As g is the inverse of f,

$\therefore f(g(x)) = x$

$\therefore f'(g(x)) \cdot g'(x) = 1$

$\therefore \frac{1}{1+(g(x))^3} \cdot g'(x) = 1$

$\therefore g'(x) = 1 + (g(x))^3$

Ans. : (C)

(116) If $g : (-\infty, \infty) \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $g(x) = 2 \tan^{-1}(e^x) - \frac{\pi}{2}$ and f is inverse of g, then $f'(0) =$ .....

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : Here, $f = g^{-1}$. So, $g^{-1}(x) = f(x)$

$g(x) = y = 2 \tan^{-1}(e^x) - \frac{\pi}{2}$

$\therefore \tan^{-1}(e^x) = \frac{\pi}{4} + \frac{y}{2}$

$\therefore x = \log \tan \left(\frac{\pi}{4} + \frac{y}{2} \right)$

$\therefore g^{-1}(y) = \log \tan\left(\frac{\pi}{4}+\frac{y}{2}\right)$ $(g(x) = y)$

$\therefore g^{-1}(x) = \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$\therefore f(x) = \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$\therefore f'(x) = \frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} \sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)\cdot\frac{1}{2}$

$\therefore f'(0) = \frac{1}{\tan\frac{\pi}{4}} \sec^2\frac{\pi}{4}\cdot\frac{1}{2} = 1$ **Ans. :** (A)

(117) If $f(x) = \lim_{n\to\infty}[(1+x)(1+x^2)(1+x^4) \ldots n \text{ terms}]$, then $f\left(\frac{1}{2}\right) = \ldots\ldots$ $|x| < 1$

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $f(x) = \lim_{n\to\infty}\left\{\frac{(1-x)(1+x)(1+x^2)(1+x^4)\ldots n \text{ terms}}{1-x}\right\}$

$= \lim_{n\to\infty}\frac{1-x^{2n}}{1-x} = \frac{1}{1-x}$ $\left(\lim_{n\to\infty} x^n = 0, |x| < 1\right)$

$\therefore f\left(\frac{1}{2}\right) = \frac{1}{1-\frac{1}{2}} = 2$ **Ans. :** (B)

(118) If $y^3\frac{d^2y}{dx^2} + k = 0$ for $y = \sqrt{2x-x^2}$, then $k = \ldots\ldots$

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $y = \sqrt{2x-x^2}$

$\frac{dy}{dx} = \frac{2-2x}{2\sqrt{2x-x^2}} = \frac{1-x}{y}$

$\therefore y\frac{dy}{dx} = 1 - x$

$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = -1$

$\therefore y^3\frac{d^2y}{dx^2} = -y^2\left(\frac{dy}{dx}\right)^2 - y^2 = -(1-x)^2 - (2x-x^2)$

$\therefore y^3\frac{d^2y}{dx^2} + 1 = 0.$ So, $k = 1$ **Ans. :** (A)

(119) If $f(x) = |x|$, $g(x) = \sin x$ and $h(x) = g(x)\, f(g(x))$, then .....

(A) $h(x)$ is continuous but not differentiable at $x = 0$.

(B) $h(x)$ is continuous and differentiable

(C) $h(x)$ is continuous and differentiable at $x = 0$ only

(D) $h(x)$ is a discontinuous function

Sol. : $h(x) = \sin x\, |\sin x| = \begin{cases} \sin^2 x\ , & x \geq 0 \qquad (0 \leq x \leq \frac{\pi}{2}) \\ -\sin^2 x, & x < 0 \qquad (-\frac{\pi}{2} \leq x < 0) \end{cases}$

Clearly $h(x)$ is a continuous and differentiable function. **Ans. :** (B)

(120) For a function $f : \mathrm{R} \to \mathrm{R}$, $f(a) = 1$, $f'(a) = 2$. If $\lim\limits_{x \to 0} \left(\dfrac{f^2(a+x)}{f(a)} \right)^{\frac{1}{x}} = e^k$, then k = ...

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $\lim\limits_{x \to 0} \left(\dfrac{f^2(a+x)}{f(a)} \right)^{\frac{1}{x}} = \lim\limits_{e^{x \to 0}} \left(\dfrac{f^2(a+x) - f(a)}{x\, f(a)} \right)$

$= \lim\limits_{e^{x \to 0}} \left(\dfrac{f^2(a+x) - f^2(a)}{x} \right)$ $(f(a) = 1)$

$= e^{2f(a)f'(a)} = e^4 = e^k$

$\therefore\ k = 4$ **Ans. :** (D)

(121) Which of the following statement is true (T) or false (F) for the function $f : \mathrm{R} \to \mathrm{R}$?

(i) If $|f(x) - f(y)| \leq 30\, |x - y|$, $\forall\ x, y \in \mathrm{R}$, then f is a continuous function on R.

(ii) If $|f(x) - f(y)| \leq 30\, |x - y|$, $\forall\ x, y \in \mathrm{R}$, then f is differentiable on R.

(iii) If $|f(x) - f(y)| \leq 21\, |x - y|^2$, $\forall\ x, y \in \mathrm{R}$, then f is differentiable on R.

(iv) If $|f(x) - f(y)| \leq 21\, |x - y|^2$, $\forall\ x, y \in \mathrm{R}$, then f is a constant function.

(A) TTTT (B) TFTT (C) FTTF (D) TFTF

Sol. : Statement (i) is true. By definition of limit (taking $\delta = \dfrac{\varepsilon}{30}$) it is easily proved.

Statement (ii) is false. We can verify by taking $f(x) = |x|$.

For statement (iii), (iv) we can have $\left| \dfrac{f(x) - f(a)}{x - a} \right| \leq 21\, |x - a|$ for some $a \in \mathrm{R}$.

$\therefore\ f$ is a differentiable function on R.

If $x \to a$ then $|f'(a)| \leq 0$

$\therefore\ |f'(a)| = 0$

$\therefore\ f$ is a constant function. So, Statement (iii), (iv) are true. **Ans. :** (B)

(122) f is a differentiable function such that $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$, $x \neq 0$, $y \neq 0$, $f(y) \neq 0$. If $f'(1) = 2$, then $f'(x) =$ .....

(A) $2f(x)$ (B) $\frac{f(x)}{x}$ (C) $2x f(x)$ (D) $\frac{2f(x)}{x}$

Sol. : Here, $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$

$\therefore f\left(\frac{1}{1}\right) = \frac{f(1)}{f(1)} = 1.$ So, $f(1) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \left\{\frac{\frac{f(x+h)}{f(x)} - 1}{h}\right\}$$

$$= f(x) \lim_{h\to 0} \frac{f\left(\frac{x+h}{x}\right)-1}{h}$$

$$= \frac{f(x)}{x} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)- f(1)}{\frac{h}{x}} = \frac{f(x)}{x} f'(1) = \frac{2f(x)}{x} \qquad (f'(1) = 2)$$

Ans. : (D)

(123) If $f(x) = \lim_{n\to\infty} \left(\sqrt[n]{x} - 1\right) n$, then $\frac{1}{f'(2012)}, \frac{1}{f'(2013)}, \frac{1}{f'(2014)}$ are in = ...

(A) arithmetic progression (B) geometric progression

(C) harmonic progression (D) none of these

Sol. : $f(x) = \lim_{n\to\infty} \frac{x^{\frac{1}{n}} - 1}{\frac{1}{n}} = \log_e x.$ So, $f'(x) = \frac{1}{x}$

$\therefore$ $\frac{1}{f'(2012)}, \frac{1}{f'(2013)}, \frac{1}{f'(2014)}$ are 2012, 2013, 2014 respectively and they are in arithmetic progression.

Ans. : (A)

(124) If $a < b < c$, $f'(x)$ is strictly increasing function on (a, c) and continuous on $[a, c]$, then .....

(A) $(b-a) f(c) + (c-b) f(a) > (c-a) f(b)$ (B) $(b-a) f(c) + (c-b) f(a) < (c-a) f(b)$

(C) $(b-a) f(a) + (c-b) f(b) > (c-a) f(c)$ (D) $(b-a) f(a) + (c-b) f(b) > (c-a) f(c)$

Sol. : Applying mean value theorem on $[a, b]$ and $[b, c]$ we have,

$\frac{f(b)-f(a)}{b-a} = f'(\alpha)$ and $\frac{f(c)-f(b)}{c-b} = f'(\beta)$, $\alpha \in (a, b)$, $\beta \in (b, c)$

Moreover, f' is strictly increasing function. So, $f'(\beta) > f'(\alpha)$

$\therefore \frac{f(c)-f(b)}{c-b} > \frac{f(b)-f(a)}{b-a}$

$\therefore (b-a)\,f(c) + (c-b)\,f(a) > (c-a)\,f(b)$ **Ans. :** (A)

(125) If for function $f : (0, \infty) \to (0, \infty)$,

(1) $f(ab) = f(a)\,f(b)$ and

(2) $\lim\limits_{x\to\infty} f(x) = c$, (where $c \neq 0$), then $f(4)$ = ...

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $\lim\limits_{x\to\infty} f(x) = c,\ c \neq 0$

Let $a \in (0, \infty)$

$c = \lim\limits_{x\to\infty} f(ax) = \lim\limits_{x\to\infty} f(a)\,f(x)$

$= f(a) \lim\limits_{x\to\infty} f(x)$

$\therefore c = f(a)\,c$

$\therefore f(a) = 1$ $(a \in (0, \infty)), c \neq 0$

$\therefore f(4) = 1$ **Ans. :** (A)

(126) Let A and B be non-empty sets. Let $f : A \to B$ and $g : B \to A$ be two functions such that $f(g(x)) = x,\ \forall\ x \in B$. Which of the following statements are true (T) or false (F) ?

(i) The function f is one-one function (ii) The function f is onto

(iii) The function g is one-one (iv) The function g is onto

(A) FTTF (B) FTFT (C) TTFF (D) TFTT

Sol. : For $x_1, x_2 \in B$, $g(x_1) = g(x_2) \Rightarrow f(g(x_1)) = f(g(x_2)) \Rightarrow x_1 = x_2 \Rightarrow g$ is one-one function.

Moreover, it is easily seen that f is onto.

(Since f is left inverse it cannot be one-one.) **Ans. :** (A)

(127) If $3f(x) - 2f\left(\frac{1}{x}\right) = x$, then $f'(2)$ = ...

(A) $\frac{2}{7}$ (B) $\frac{1}{2}$ (C) 2 (D) $\frac{7}{2}$

Sol. : $3f(x) - 2f\left(\frac{1}{x}\right) = x$ (1)

Replacing x by $\frac{1}{x}$ we get, $-2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1}{x}$ (2)

From (1) and (2) eliminate $f\left(\frac{1}{x}\right)$ we get, $f(x) = \frac{1}{5}\left(3x + \frac{2}{x}\right)$

$\therefore f'(x) = \frac{1}{5}\left(3 - \frac{2}{x^2}\right)$. So, $f'(2) = \frac{1}{2}$ **Ans. :** (B)

(128) If $v e^{\frac{u}{v^3}} = 1$, then .....

(A) $v \frac{d^2u}{dv^2} + 2 \frac{du}{dv} + 3v^2 = 0$ (B) $v \frac{d^2u}{dv^2} - 2 \frac{du}{dv} + 3v^2 = 0$

(C) $v \frac{d^2u}{dv^2} + 2 \frac{du}{dv} = 3v^2$ (D) $\frac{d^2u}{dv^2} - 2 \frac{du}{dv} = 3v^3$

Sol. : $v e^{\frac{u}{v^3}} = 1$

$\log v + \frac{u}{v^3} = 0.$ So, $u + v^3 \log v = 0$ (1)

$\therefore \frac{du}{dv} + v^3 \frac{1}{v} + 3v^2 \log v = 0$

$\therefore v \frac{du}{dv} + v^3 + 3v^3 \log v = 0$

$\therefore v \frac{du}{dv} + v^3 + 3 (-u) = 0$ (From (1))

$\therefore v \frac{d^2u}{dv^2} + \frac{du}{dv} + 3v^2 - 3 \frac{du}{dv} = 0$

$\therefore v \frac{d^2u}{dv^2} - 2 \frac{du}{dv} + 3v^2 = 0$ **Ans. :** (B)

(129) $f : \mathrm{R} \to \mathrm{R}$ is a differentiable function. If $f(y) f(x - y) = f(x)$, $\forall x, y \in \mathrm{R}$ and $f'(0) = p$, $f'(5) = q$, $p, q \neq 0$, then $f'(-5) =$ .....

(A) $\frac{p^2}{q}$ (B) $\frac{p}{q}$ (C) $\frac{q}{p}$ (D) q

Sol. : Differentiate $f(y) f(x - y) = f(x)$ with respect to x and we get, $f(y) f'(x - y) = f'(x)$

Putting $x = y$ we get, $f(x) f'(0) = f'(x)$

$\therefore f'(x) = p f(x)$ $(f'(0) = p)$ (1)

$\therefore \int \frac{f'(x)}{f(x)} dx = \int p \, dx + c$

$\therefore \log_e f(x) = px + c$ (2)

Putting $x = 0$ in equation (1) we get, $f'(0) = p f(0)$. So, $p = pf(0)$. So, $f(0) = 1$

By equation (2) we get, $\log_e f(0) = c$

$\therefore \log_e 1 = c.$ Hence, $c = 0$

$\therefore \log_e f(x) = px$

$\therefore f(x) = e^{px}.$ So, $f'(x) = pe^{px}$

$\therefore f'(5) = pe^{5p} = q.$ So, $e^{5p} = \frac{q}{p}$

$\therefore f'(-5) = pe^{-5p} = \frac{p}{q/p} = \frac{p^2}{q}$ **Ans. :** (A)

(130) $\lim_{n\to\infty}\left(\sqrt[3]{n^2-n^3}+n\right)$ = ...

(A) $\frac{1}{3}$ (B) $-\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $-\frac{2}{3}$

Sol. : $\lim_{n\to\infty}\left(\sqrt[3]{n^2-n^3}+n\right) = \lim_{n\to\infty} n\left[\left(\frac{1}{n}-1\right)^{\frac{1}{3}}+1^{\frac{1}{3}}\right]$

$= \lim_{n\to\infty} n\left[\frac{\left(\frac{1}{n}-1\right)+1}{\left(\frac{1}{n}-1\right)^{\frac{2}{3}}-\left(\frac{1}{n}-1\right)^{\frac{1}{3}}+1}\right]$ $\left(a+b=\frac{a^3+b^3}{a^2-ab+b^2}\right)$

$= \lim_{n\to\infty}\left[\frac{1}{\left(\frac{1}{n}-1\right)^{\frac{2}{3}}-\left(\frac{1}{n}-1\right)^{\frac{1}{3}}+1}\right] = \frac{1}{3}$ **Ans. :** (A)

(131) If the tangent to the curve $xy + ax + by = 0$ at (1, 1) makes an angle of measure $tan^{-1}\ 2$ with positive X-axis, then $\frac{a+b}{ab}$ = ...

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) 1

Sol. : Differentiating $xy + ax + by = 0$, we get, $\left(x\frac{dy}{dx}+y+a+b\frac{dy}{dx}\right) = 0$

$\therefore \frac{dy}{dx} = -\left(\frac{a+y}{b+x}\right)$

$\therefore \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{a+1}{b+1} = 2$ (Given tangent makes an angle of measure $tan^{-1}\ 2$ with positive X-axis.)

$\therefore a + 2b = -3$ (1)

Moreover, (1, 1) lies on the curve. So, $1 + a + b = 0$. Hence, $a + b = -1$ (2)

Solving (1) and (2) we get, $a = 1$, $b = -2$

$\therefore \frac{a+b}{ab} = \frac{1}{2}$ **Ans. :** (B)

(132) If the rate of increase of a side of a triangle inscribed in a circle of radius R, is R times the rate of increase of the measure of opposite angle, then the measure of the angle is ..... .

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Sol. : Let the side of the triangle be BC = a and angle opposite to it is A.

$\therefore a = 2R\ sin\ A$

$\frac{da}{dt} = 2R\ cos\ A\ \frac{dA}{dt}$. So, $R\frac{dA}{dt} = 2R\ cos\ A\ \frac{dA}{dt}$ $\left(\frac{da}{dt} = R\frac{dA}{dt}\right)$

$\therefore\ cosA = \frac{1}{2}$. Hence, $A = \frac{\pi}{3}$ **Ans. :** (C)

(133) $f(x) = sin\ x + cos\ x$, $0 \le x \le 2\pi$ is strictly decreasing in ..... interval.

(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ (C) $\left(\frac{5\pi}{4}, 2\pi\right)$ (D) $(0, 2\pi)$

Sol. : $f(x) = sin\ x + cos\ x$

$f'(x) = cos\ x - sin\ x$

Now, $f'(x) = 0 \Rightarrow sin\ x = cos\ x$. So, $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ $(0 \le x \le 2\pi)$

Interval	$f'(x)$	Decision about f
$\left(0, \frac{\pi}{4}\right)$	Positive	f strictly increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	Negative	f strictly decreasing
$\left(\frac{5\pi}{4}, 2\pi\right)$	Positive	f strictly increasing

$\therefore$ f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. **Ans. :** (B)

(134) The sum of the perimeters of a circle and a square is constant. When the sum of their areas is minimum, then the ratio of the side of the square to the radius of the circle is ..... .

(A) 1 : 2 (B) 2 : 1 (C) 1 : 3 (D) 4 : 1

Sol. : Let the radius of the circle be r and the side of the square be x. The sum of their perimeters is

$2\pi r + 4x = 2k$, k constant

The sum of areas $= \pi r^2 + x^2$, call it $f(r)$.

$\therefore f(r) = \pi r^2 + \frac{(k - \pi r)^2}{4}$

$f'(r) = 2\pi r - 2\pi\frac{(k - \pi r)}{4} = 0 \Rightarrow k = \pi r + 4r$

$\therefore\ 2\pi r + 4x = 2\pi r + 8r$ $(2k = 2\pi r + 4x)$

Hence, $x = 2r$ So, $\frac{x}{r} = \frac{2}{1}$

Moreover, $f''(r) = 2\pi - \frac{\pi}{2}(0 - \pi) = 2\pi + \frac{\pi^2}{2} > 0$

$\therefore$ When the ratio of side of the square to radius of the circle is 2:1 the sum of their areas is minimum. **Ans. :** (B)

(135) If tangents at some point (x, y) of the curves $y = x\ cos\ x$ and $y = \frac{sinx}{x}$ are parallel to X-axis, then x is a root of ..... equations.

(A) $sin\ x = x$, $tan\ x = x$ (B) $cot\ x = x$, $sec\ x = x$

(C) $cot\ x = x$, $tan\ x = x$ (D) $tan\ x = x$, $sec\ x = x$

Sol. : Let $f(x) = x \cos x$ and $g(x) = \frac{\sin x}{x}$

$f'(x) = -x \sin x + \cos x$ and $g'(x) = \frac{x\cos x - \sin x}{x^2}$

Now, if tangents are parallel to X-axis, then $f'(x) = 0$ and $g'(x) = 0$.

$\therefore \cot x = x$ and $\tan x = x$ (Why $\sin x = 0$, $\cos x = 0$ not possible ?) **Ans. :** (C)

(136) The equation $e^{x-8} + 2x - 17 = 0$ has .....

(A) two real roots (B) one real root

(C) eight real roots (D) infinitely many solutions

Sol. : Clearly $x = 8$ is one of the solutions of the given equation. Let if possible $\alpha \neq 8$ be another solution of given equation. Let $\alpha > 8$. Applying Rolles' theorem in the interval $[8, \alpha]$, there exists $c \in (8, \alpha)$ such that $f'(c) = 0$.

$\therefore e^{c-8} + 2 = 0$. So, $e^{c-8} = -2$ which is not possible. If $\alpha < 8$, then also we get a similar result.

$\therefore$ Given equation does not have any other solution except $x = 8$. **Ans. :** (B)

Second method :

$f'(x) = e^{x-8} + 2 > 0, \ \forall \ x$

$\therefore f$ is an increasing function.

$\therefore f$ is one-one function.

$\therefore f$ does not have two or more zeros.

In the first method for $\alpha < 8$, f does not have two roots as f is increasing.

(137) If the curves $x^2y = 1$ and $x^3 = a^5y$ intersect at right angles, then $a^6 =$ .....

(A) 2 (B) 4 (C) 6 (D) 8

Sol. : First of all, we find points of intersection of given curves.

$x^2y = 1$ and $x^3 = a^5y$

$\therefore x^2\left(\frac{x^3}{a^5}\right) = 1$. So, $x^5 = a^5$. Hence, $x = a$

Since $y = \frac{x^3}{a^5}$, we have $y = \frac{1}{a^2}$. Thus, the point of intersection is $P\left(a, \frac{1}{a^2}\right)$.

For the curve $x^2y = 1$, we have $2xy + x^2\frac{dy}{dx} = 0$. So, $\frac{dy}{dx} = -\frac{2y}{x}$ $(x \neq 0)$

$\therefore \left(\frac{dy}{dx}\right)_{x=a} = -\frac{2}{a^3} = m_1$

For the curve $x^3 = a^5y$, we have $3x^2 = a^5\frac{dy}{dx}$. So, $\frac{dy}{dx} = \frac{3x^2}{a^5}$. So, $\left(\frac{dy}{dx}\right)_{x=a} = \frac{3}{a^3} = m_2$

Since given curves intersect at right angles, $m_1 m_2 = -1$.

Hence, $\frac{-6}{a^6} = -1$. So, $a^6 = 6$ **Ans. :** (C)

(138) Let a function f be twice differentiable and $f''(t) < 0$, $\forall\, t \in R$. If $f'(y) + x < f(y+1)$; $x, y \in R$, then .....

(A) $f(y) = x$ (B) $f(y) > x$ (C) $f(y) < x$ (D) $f(y) \le x$

Sol. : Applying mean value theorem to f in $[y, y+1]$, we have $f'(c) = \frac{f(y+1)-f(y)}{y+1-y}$, $c \in (y, y+1)$

$f''(t) < 0$. So, f' is decreasing.

Also, $y < c < y+1$

$\therefore f'(c) < f'(y)$

$\therefore f(y+1) - f(y) < f'(y)$

$\therefore x + f'(y) < f(y+1) < f'(y) + f(y)$

$\therefore x < f(y)$ **Ans. :** (B)

(139) $\lim\limits_{x\to 0} \dfrac{x^2 \int\limits_0^x \sqrt{t^3+1}\,dt}{\int\limits_0^x t^2 \cos t\,dt} =$ .....

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $\lim\limits_{x\to 0} \dfrac{x^2 \int\limits_0^x \sqrt{t^3+1}\,dt}{\int\limits_0^x t^2 \cos t\,dt}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{x^2\sqrt{x^3+1} + 2x\int\limits_0^x \sqrt{t^3+1}\,dt}{x^2 \cos x}$

$= \lim\limits_{x\to 0} \dfrac{\sqrt{x^3+1}}{\cos x} + 2\lim\limits_{x\to 0} \dfrac{\int\limits_0^x \sqrt{t^3+1}\,dt}{x\cos x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= 1 + 2 \lim\limits_{x\to 0} \dfrac{\sqrt{x^3+1}}{\cos x - x\sin x} = 1 + 2 = 3$ **Ans. :** (D)

(140) Let $f(x)$ be a differentiable function and G is the graph of $f(x)$. If P$(a, f(a))$ on G is the point nearest to (0, 0), then $f(a)\, f'(a) =$.....

(A) a (B) $-a$ (C) 1 (D) -1

Sol. : Distance function $D(x) = x^2 + (f(x))^2$ (For maximum/minimum no need to take square root)

$\therefore D'(x) = 2x + 2 f(x) f'(x)$

Since OP is minimum, we have $D'(a) = 0$

$\therefore a + f(a) f'(a) = 0$

$\therefore f(a) f'(a) = -a$ (Verify $D''(x) > 0$) **Ans. :** (B)

(141) On the curve $y^2 - x^2 = 1$ there is a point P whose x coordinate is n where $n \in N$. If d_n represents the distance from P to the line $y = x$, then $\lim\limits_{n\to\infty} (nd_n)$ = ...

(A) 1 (B) $2\sqrt{2}$ (C) $\frac{1}{2\sqrt{2}}$ (D) $\frac{1}{2}$

Sol. : Let P(n, $\sqrt{n^2+1}$) be any point on the curve $y^2 - x^2 = 1$, where $n \in$ N

$$d_n = \left|\frac{\sqrt{n^2+1}-n}{\sqrt{2}}\right| = \frac{\sqrt{n^2+1}-n}{\sqrt{2}}$$

$$\therefore \lim_{n\to\infty} n\cdot d_n = \frac{1}{\sqrt{2}} \lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right)$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{n\left(n^2+1-n^2\right)}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2\sqrt{2}}$$ **Ans. :** (C)

(What if the point is P(n, $-\sqrt{n^2+1}$) ?)

(142) Let $F(x) = \begin{cases} \left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}} & , x \neq 0 \\ e^3 & , x = 0 \end{cases}$ and $G(x) = \begin{cases} \left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}} & , x \neq 0 \\ k & , x = 0 \end{cases}$

where $f(x)$ is a real function. If $F(x)$ is continuous at $x = 0$ and $G(x)$ is continuous at $x = 0$, then what is the value of log k ?

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $F(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x\to0} F(x) = F(0) = e^3$$

$$\therefore \lim_{x\to0} \left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}} = e^3$$

$$\therefore \lim_{x\to0} e^{\frac{1}{x}\left(x+\frac{f(x)}{x}\right)} = e^3. \qquad \text{So, } \lim_{x\to0} e^{1+\lim_{x\to0}\frac{f(x)}{x^2}} = e^3$$

$$\therefore \lim_{x\to0} \frac{f(x)}{x^2} = 2 \qquad (1)$$

Now, $G(x)$ is continuous at $x = 0$.

So, $\lim_{x\to0} G(x) = k$.

$$\therefore \lim_{x\to0} \left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}} = k. \qquad \text{So, } e^{\lim_{x\to0}\frac{1}{x}\left(\frac{f(x)}{x}\right)} = k$$

$$\therefore e^{\lim_{x\to0}\frac{f(x)}{x^2}} = k. \qquad \text{So, } e^2 = k \qquad \text{Hence, } \log k = 2$$ **Ans. :** (C)

(143) Let, $f(x) = x^2 - bx + c$, where b, c are odd natural numbers and $f(x) = 0$ has both the roots primes. If $b + c = 35$, then the minimum value of $f(x)$ is ..... .

(A) $-\frac{183}{4}$ (B) $\frac{173}{16}$ (C) $-\frac{81}{4}$ (D) $\frac{81}{4}$

Sol. : Let, α, β be roots of $f(x) = 0$

$\therefore \alpha + \beta = b$

α, β are primes and b is odd natural number. We have to take $\alpha = 2$. $f(2) = 0$.

$\therefore 2b - c = 4$ (1)

$\therefore b + c = 35$ (2)

$\therefore b = 13,\ c = 22$

$f'(x) = 2x - b$ So, $f'(x) = 0 \Rightarrow x = \frac{b}{2} = \frac{13}{2}$

$\therefore f''(x) = 2 > 0$

$\therefore$ The minimum value of f is $f\left(\frac{13}{2}\right) = -\frac{81}{4}$

Remark : The minimum value is $\frac{4ac-b^2}{4a} = \frac{88-169}{4} = -\frac{81}{4}$ **Ans. :** (C)

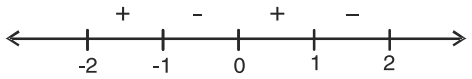
(144) At how many points does the function $f(x) = \begin{cases} (x+1)^3 & , -2 < x \le -1 \\ x^{\frac{2}{3}} - 1 & , -1 < x \le 1 \\ -(x-1)^2 & , 1 < x < 2 \end{cases}$ have real maximum and minimum values ?

(A) 4 (B) 3 (C) 2 (D) 1

Sol. : $f'(x) = \begin{cases} 3(x+1)^2, -2 < x < -1 \\ \frac{2}{3}x^{-\frac{1}{3}} , x \in (-1, 1) - \{0\} \\ -2(x-1), 1 < x < 2 \end{cases}$

f' is positive in (-2, -1)
f' is negative in (-1, 0)
f' is positive in (0, 1)
f' is negative in (1, 2)

f' does not exist at the points $x = -1, 0, 1$.



$\therefore$ f has local maximum and minimum at three points. **Ans. :** (B)

(145) The measure of the angle between the curves $y = [\,|\sin x| + |\cos x|\,]$ and $x^2 + y^2 = 5$ is ..... . where $[.]$ is an integer part function.

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\tan^{-1} 3$ (D) $\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Sol. : $y = [\,|\sin x| + |\cos x|\,] = 1.\ \forall\ x \in R$ $(1 \le |\sin x| + |\cos x| \le \sqrt{2})$

$\therefore$ Given curves intersect at the points P(2, 1) and Q(–2, 1). For the first curve $y = 1$, we have $\frac{dy}{dx} = 0 = m_1$ and for $x^2 + y^2 = 5$, we have $\frac{dy}{dx} = -\frac{x}{y}$.

$\therefore$ For point P(2, 1), $\left(\frac{dy}{dx}\right)_{(2,1)} = -2 = m_2$

Let θ be the measure of the angle between given curves at P.

$\therefore \tan\theta = \left|\frac{0+2}{1+0}\right| = 2$

$\therefore \theta = \tan^{-1} 2 = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$

At point Q also we get $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$. **Ans. :** (D)

(146) If $a, b > 0$, then the minimum value $y = \frac{b^2}{a-x} + \frac{a^2}{x}$, $0 < x < a$ is ..... .

(A) $\frac{a+b}{a}$ (B) $\frac{ab}{a+b}$ (C) $\frac{1}{a}+\frac{1}{b}$ (D) $\frac{(a+b)^2}{a}$

Sol. : $y = \frac{b^2}{a-x} + \frac{a^2}{x}$

$\frac{dy}{dx} = \frac{-b^2}{(a-x)^2}(-1) - \frac{a^2}{x^2}$

$\frac{dy}{dx} = 0 \Rightarrow bx = \pm a(x-a) = \pm(ax - a^2)$

$\therefore x = \frac{a^2}{a \pm b}$

Since $0 < x < a$, we have $x = \frac{a^2}{a+b}$ and $a - x = \frac{ab}{a+b}$

$\frac{d^2y}{dx^2} = \frac{2a^2}{x^3} + \frac{2b^2}{(a-x)^3}$. So, $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{a^2}{a+b}} > 0$

$\therefore y$ is minimum at $x = \frac{a^2}{a+b}$

The minimum value of y is $\frac{b^2(a+b)}{ab} + \frac{a^2(a+b)}{a^2} = (a+b)\left[\frac{b}{a}+1\right] = \frac{(a+b)^2}{a}$ **Ans. :** (D)

(147) If α, β and γ are the roots of the equation $x^3 + x^2 - 5x - 1 = 0$, then $[\alpha] + [\beta] + [\gamma]$ = ...

(A) 2 (B) –3 (C) 4 (D) 5

Sol. : Let $f(x) = x^3 + x^2 - 5x - 1$

$f'(x) = 3x^2 + 2x - 5 = (x-1)(3x+5)$

If $x \in \left(-\infty, -\frac{5}{3}\right) \cup (1, \infty)$, then $f'(x) > 0$

and If $x \in \left(-\frac{5}{3}, 1\right)$, then $f'(x) < 0$

$\therefore f$ is increasing in $(-\infty, -\frac{5}{3})$, decreasing in $(-\frac{5}{3}, 1)$ and increasing in $(1, \infty)$.

Moreover, $f\left(-\frac{5}{3}\right) = \frac{128}{27}$, $f(1) = -4$ and $f(0) = -1$. Rough sketch of $f(x)$ would be like this.

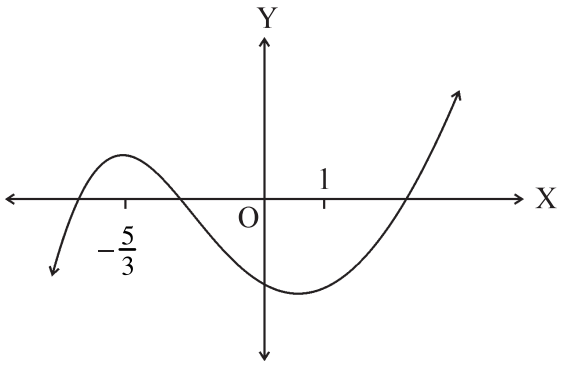


Figure 15.5

$f(0) = -1 < 0$. $f(1) = -4 < 0$, $f(2) = 1 > 0$

$\therefore$ One root lies in (1, 2). So, $[\alpha] = 1$

$f(-1) = 4 > 0$

$\therefore$ One root lies in (–1, 0). So, $[\beta] = -1$

$f(-2) = 5 > 0$

$f(-3) = -4 < 0$

$\therefore$ One root lies in (–3, –2). So, $[\gamma] = -3$

$\therefore$ $[\alpha] + [\beta] + [\gamma] = -3$ **Ans. :** (B)

(148) For the curves $x = a\sqrt{cos2\theta}\ cos\theta$, $y = a\sqrt{cos2\theta}\ sin\theta$, the tangent at $\theta = \frac{\pi}{6}$ is .....

(A) parallel to X-axis or X-axis itself. (B) parallel to Y-axis or Y-axis itself.

(C) parallel to the line $y = x$ (D) parallel to the line $x + y = 1$

Sol. : $\frac{dx}{d\theta} = -a\sqrt{cos2\theta}\ sin\theta + \left(\frac{-a\,cos\theta\ sin2\theta}{\sqrt{cos2\theta}}\right)$

$= -a\frac{(cos2\theta\ sin\theta + cos\theta\ sin2\theta)}{\sqrt{cos2\theta}} = \frac{-asin3\theta}{\sqrt{cos2\theta}}$

$\frac{dy}{d\theta} = a\sqrt{cos2\theta}\ cos\theta + \left(\frac{-a\,sin\theta\ sin2\theta}{\sqrt{cos2\theta}}\right) = \frac{a\,cos3\theta}{\sqrt{cos2\theta}}$

$\therefore \frac{dy}{dx} = -cot3\theta$ So, $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = 0$

$\therefore$ The tangent is parallel to X-axis or it is X-axis. **Ans. :** (A)

(149) If $f(x) = \frac{x}{sinx}$ and $g(x) = \frac{x}{tanx}$, $0 < x < 1$, then in this interval

(A) $f(x)$ and $g(x)$ both are increasing (B) $f(x)$ and $g(x)$ both are decreasing

(C) $f(x)$ is increasing and $g(x)$ is decreasing (D) $g(x)$ is increasing and $f(x)$ is decreasing

Sol. : $f'(x) = \frac{sinx - xcosx}{sin^2x}$, $g'(x) = \frac{tanx - xsec^2x}{tan^2x}$

Let $u(x) = sinx - xcosx$

$\therefore$ $u'(x) = xsinx > 0$ $(0 < x < 1)$

$\therefore$ u is increasing on (0, 1). Also $u(0) = 0$

So, $u(x) > 0$. Hence, $sinx > x\ cos\ x$

$\therefore f'(x) > 0$ $(0 < x < 1)$

$\therefore f$ is increasing on (0, 1).

Let, $v(x) = tanx - xsec^2x$

$\therefore v'(x) = -2x\ sec^2x\ tanx < 0$. $v(x)$ is $\downarrow$. $v(0) = 0$. $(0 < x < 1)$

$\therefore g'(x) < 0$

$\therefore g$ is decreasing on (0, 1). **Ans. :** (C)

(150) Function $g(x) = tanx\ .\ e^x\ e^{x^2}\ \frac{\log(\pi + x)}{\log(e + x)}$ $(x \geq 0)$ is ..... , $x \neq (2k + 1)\frac{\pi}{2}$, $k \in Z$

(A) increasing in $[0, \infty)$

(B) decreasing in $[0, \infty)$

(C) increasing in $\left[0, \frac{\pi}{e}\right]$ and decreasing in $\left(\frac{\pi}{e}, \infty\right)$

(D) decreasing in $\left[0, \frac{\pi}{e}\right]$ and increasing in $\left(\frac{\pi}{e}, \infty\right)$

Sol. : $tanx$, e^x, e^{x^2} are increasing in $[0, \infty)$. (*tan* whenever defined)

Let $f(x) = \frac{\log(\pi + x)}{\log(e + x)}$

$$f'(x) = \frac{(e + x)\log(e + x) - (\pi + x)\log(\pi + x)}{(\pi + x)(e + x)\left(\log(e + x)\right)^2}$$

log function is increasing.

$\therefore e < \pi \Rightarrow \log(e + x) < \log(\pi + x)$

$\therefore (e + x)\log(e + x) < (e + x)\log(\pi + x) < (\pi + x)\log(\pi + x)$, $\forall\ x \geq 0$

$\therefore f'(x) < 0$, $\forall\ x \geq 0$

$\therefore f$ is decreasing in $[0, \infty)$.

$\therefore g$ is decreasing in $[0, \infty)$. **Ans. :** (B)

(151) In which interval does a lie, if $f(x) = sinx - asin2x - \frac{1}{3}sin3x + 2ax$ is increasing on R ?

(A) $[0, \infty)$ (B) $(0, \infty)$ (C) $\left[\frac{1}{2}, \infty\right)$ (D) $[1, \infty)$

Sol. : $f'(x) = cosx - 2acos2x - cos3x + 2a$

$= cosx - 2a(2cos^2x - 1) - (4cos^3x - 3cosx) + 2a$

$= 4a + 4cosx - 4acos^2x - 4cos^3x$

$= 4asin^2x + 4cosxsin^2x$

$= 4sin^2x\ (a + cosx)$

$-1 \leq cosx \leq 1 \Rightarrow a - 1 \leq a + cosx \leq a + 1$. If $a \geq 1$, then $f'(x) \geq 0$.

$\therefore$ If $a \in [1, \infty)$, then f is increasing on R. **Ans. :** (D)

(152) Which point is the farthest point from (0, –2) lying on the curve $4x^2 + a^2y^2 = 4a^2$? $4 < a^2 < 8$

(A) (0, 1) (B) (0, –1) (C) (0, 2) (D) (0, –2)

Sol. : For the curve $\frac{x^2}{a^2}+\frac{y^2}{4}=1$ a parametric point is P($a\cos\theta$, $2\sin\theta$). The square of the distance from point (0, –2) is $r^2 = (a\cos\theta - 0)^2 + (2\sin\theta + 2)^2 = a^2\cos^2\theta + 4(1 + \sin\theta)^2$.

$$\frac{d(r)^2}{d\theta} = -2a^2\sin\theta\cos\theta + 8(1 + \sin\theta)\cos\theta$$

$$= \cos\theta\{(8 - 2a^2)\sin\theta + 8\}$$

$$\frac{d(r)^2}{d\theta} = 0 \Rightarrow \cos\theta = 0 \text{ or } \sin\theta = \frac{4}{a^2 - 4}$$

Since $4 < a^2 < 8$, we have $\frac{4}{a^2-4} > 1$. i.e. $\sin\theta > 1$ is not possible.

$\therefore \theta = \frac{\pi}{2}$ as $\cos\theta = 0$

Also, $\left[\frac{d^2}{d\theta^2}r^2\right]_{\theta=\frac{\pi}{2}} = (4 - a^2)2\cos 2\theta - 8\sin\theta = 2a^2 - 8 - 8 < 0$

$\therefore$ For $\theta = \frac{\pi}{2}$, r^2 is maximum.

$\therefore$ Required point is (0, 2).. **Ans. :** (C)

(153) Let an ellipse have centre at O. Let P be a point on $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. Let N be the foot of perpendicular drawn from O to the tangent at P. Let A_{max} be the largest area of ΔOPN. The value of $\frac{a^2 - b^2}{A_{max}}$ = ...

(A) 1 (B) 2 (C) 4 (D) 8

Sol. : Let the coordinates of P be ($a\cos\theta$, $b\sin\theta$).

$$\therefore \left(\frac{dy}{dx}\right)_P = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$$

The equation of the tangent at P,

$bx\cos\theta + ay\sin\theta = ab$

$$ON = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

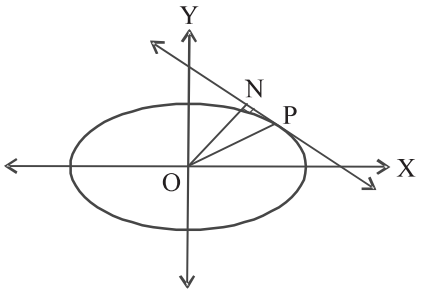


Figure 15.6

In right ΔONP,

$PN^2 = OP^2 - ON^2$

$$= a^2\cos^2\theta + b^2\sin^2\theta - \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{a^4\sin^2\theta\cos^2\theta + b^4\sin^2\theta\cos^2\theta + a^2b^2\left(\sin^4\theta + \cos^4\theta\right) - a^2b^2\left(\sin^2\theta + \cos^2\theta\right)^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{\left(a^2 - b^2\right)^2 \sin^2\theta\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\therefore \text{PN} = \frac{\left(a^2 - b^2\right)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\text{A} = \frac{1}{2}\,\text{PN} \times \text{ON} = \frac{ab\left(a^2 - b^2\right)\sin\theta\cos\theta}{2\left(a^2\sin^2\theta + b^2\cos^2\theta\right)}$$

$$= \frac{ab\left(a^2 - b^2\right)}{2\left(a^2\tan\theta + b^2\cot\theta\right)} \qquad (1)$$

For A to be maximum, $f(\theta) = a^2\tan\theta + b^2\cot\theta$ should be minimum.

$$\frac{d\left(f(\theta)\right)}{d\theta} = a^2\sec^2\theta - b^2\operatorname{cosec}^2\theta, \quad f''(\theta) = 2a^2\sec^2 a\,\tan\theta + 2b^2\operatorname{cosec}^2 a\,\cot\theta$$

$$\frac{d\left(f(\theta)\right)}{d\theta} = 0 \Rightarrow \tan\theta = \pm\frac{b}{a}$$

$$\frac{d^2\left(f(\theta)\right)}{d\theta^2}\Bigg|_{\tan\theta = \frac{b}{a}} > 0$$

$\therefore$ When $\tan\theta = \frac{b}{a}$, then $f(\theta)$ is minimum. So, minimum value of $f(\theta) = a^2\frac{b}{a} + b^2\frac{a}{b} = 2ab$

$$\therefore \text{A}_{\max} = \frac{ab\left(a^2 - b^2\right)}{2.\,2ab} = \frac{a^2 - b^2}{4}$$

$$\therefore \frac{a^2 - b^2}{\text{A}_{\max}} = 4$$ **Ans. :** (C)

(154) $f(x) = \begin{cases} xe^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$ where $a > 0$.

Let L(a) be the length of the interval in which $f'(x)$ is strictly increasing. $\frac{1}{\text{L}'(3)} = ...$

(A) 3 (B) 6 (C) 9 (D) 12

Sol. : $f'(x) = \begin{cases} (1 + ax)e^{ax}, & x < 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$

$$f''(x) = \begin{cases} (2 + ax)a\,e^{ax}, & x < 0 \\ 2a - 6x, & x > 0 \end{cases}$$

If $f'(x)$ is strictly increasing function, then we should have $f''(x) > 0$

$\therefore$ $2 + ax > 0$ $(x < 0)$ and $2a - 6x > 0$ $(x > 0)$

$\therefore x > \frac{-2}{a}$ $(x < 0)$ and $x < \frac{a}{3}$ $(x > 0)$

$\therefore \frac{-2}{a} < x < \frac{a}{3}, x \neq 0$

$\therefore$ L$(a) = \frac{a}{3} + \frac{2}{a}$. So, L'$(a) = \frac{1}{3} - \frac{2}{a^2}$

$\therefore$ L'$(3) = \frac{1}{9}$. So, $\frac{1}{\text{L'}(3)} = 9$ **Ans. :** (C)

(155) Function $2tan^3x - 3tan^2x + 12tanx + 3$ is .....

(A) increasing

(B) decreasing

(C) increasing in $\left(0, \frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(D) decreasing in $\left(0, \frac{\pi}{4}\right)$ and increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Sol. : Let $f(x) = 2tan^3x - 3tan^2x + 12tanx + 3$

$f'(x) = (6tan^2x - 6tanx + 12)\ sec^2x$

$= 6sec^2x\ (tan^2x - tanx + 2) = 6sec^2x \left[\left(tan\ x - \frac{1}{2}\right)^2 + \frac{7}{4}\right] > 0$

$\therefore f(x)$ is increasing. **Ans. :** (A)

(156) For which value of x is $f(x) = \sum_{k=1}^{2015} (x-k)^2$ minimum ?

(A) 0 (B) 1008 (C) 1000 (D) 2015

Sol. : $f'(x) = \sum_{k=1}^{2015} 2(x-k) = 2[(x-1) + (x-2) + ... + (x-2015)]$

$f'(x) = 0 \Rightarrow 2015x - \frac{(2015)(2016)}{2} = 0$

$x = 1008$. Moreover $f''(x) = 4030 > 0$

$\therefore f$ is minimum for $x = 1008$. **Ans. :** (B)

(157) What is the set of values of a for which

$f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ is decreasing on R ?

(A) $\left(-3, \frac{5-\sqrt{27}}{2}\right) \cup (2, \infty)$ (B) $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$ (C) $(-\infty, \infty)$ (D) $(1, \infty)$

Sol. : $f'(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) 5x^4 - 3$

$f'(x) \leq 0$ is necessary for $f(x)$ to be decreasing, $\forall\ x \in$ R

$\therefore \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^4 < \frac{3}{5},\ \forall\ x$ (1)

If $\frac{\sqrt{a+4}}{1-a} - 1 \leq 0$, then only (1) is possible.

$\therefore$ For $a > 1$, this inequality is true.

$\therefore a \in (1, \infty)$

Moreover, $\sqrt{a+4}$ to be real $a \geq -4$. Let us consider $a < 1$.

$\therefore \frac{\sqrt{a+4}}{1-a} \leq 1$. So, $\sqrt{a+4} \leq 1-a$

$\therefore a + 4 \leq 1 + a^2 - 2a$

$\therefore 0 \leq a^2 - 3a - 3$. So, $a \leq \frac{3-\sqrt{21}}{2}$

$\therefore a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$ **Ans. :** (B)

(158) The maximum value of the slope of tangent to the curve $y = x^3 - 6x^2 + 9x + 4$, $0 \leq x \leq 5$ is ..... .

(A) 2 (B) 3 (C) 4 (D) 24

Sol. : $m = \frac{dy}{dx} = 3x^2 - 12x + 9$

For the extreme value of m, $\frac{dm}{dx} = 0$

$\therefore 6x - 12 = 0$ So, $x = 2$. $\left[\frac{d^2m}{dx^2}\right]_{x=2} = 6$ So, for $x = 2$ we get local minimum value.

Now, $m(0) = 9$, $m(2) = -3$ and $m(5) = 24$

$\therefore$ Global maximum value of m is 24. **Ans. :** (D)

(159) Let f be differentiable on R. If $h(x) = f(x) - (f(x))^2 + (f(x))^3$, $x \in$ R then,

(A) if f is a decreasing function, then h is also decreasing

(B) if f is an increasing function, then h is decreasing

(C) if f is either increasing or decreasing function, then accordingly h is also increasing or decreasing

(D) can't say anything about h.

Sol. : $h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$= f'(x)[1 - 2f(x) + 3(f(x))^2]$

$= 3f'(x)\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$

If $f'(x) \geq 0$, then $h'(x) \geq 0$ and if $f'(x) \leq 0$, then $h'(x) \leq 0$.

$\therefore$ If the function f is increasing then h is increasing and if f is decreasing then h is decreasing.

Ans. : (C)

(160) Let the tangent at the point (a, a) of the curve $2y^3 = ax^2 + x^3$ have intercepts α, β on axes. If $\alpha^2 + \beta^2 = 61$, then $|a|$ = ...

(A) 16 (B) 28 (C) 30 (D) 31

Sol. : The slope of the tangent $\frac{dy}{dx} = \frac{2ax + 3x^2}{6y^2}$. So, $\left(\frac{dy}{dx}\right)_{(a,a)} = \frac{5}{6}$

$\therefore$ The equation of the tangent at (a, a) is

$y - a = \frac{5}{6}(x - a)$.

$\therefore\ 6y - 6a = 5x - 5a.$ So, $5x - 6y = -a$

X-intercept $\alpha = -\frac{a}{5}$, Y-intercept $\beta = \frac{a}{6}$

$\alpha^2 + \beta^2 = 61.$ $\therefore\ \frac{a^2}{25} + \frac{a^2}{36} = 61$

$\therefore\ a^2 = 25 \times 36.$ So, $a = \pm 30.$ Hence, $|a| = 30$ **Ans. :** (C)

(161) The difference between the maximum value and the minimum value of the function

$f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$ is ..... .

(A) $\frac{3}{8}$ (B) $\frac{2}{3}$ (C) $\frac{8}{7}$ (D) $\frac{9}{4}$

Sol. : The function $f(x)$ has principal period 2π so we find the required difference in $[0, 2\pi]$.

$$f'(x) = -(\sin x + \sin 2x - \sin 3x)$$
$$= -\sin x + \sin 3x - \sin 2x$$
$$= -2 \sin \frac{x}{2} \cos \frac{x}{2} + 2 \cos \frac{5x}{2} \sin \frac{x}{2}$$
$$= 2 \sin \frac{x}{2}\left(\cos\frac{5x}{2} - \cos\frac{x}{2}\right)$$
$$= -4 \sin x \sin \frac{3x}{2} \sin \frac{x}{2}$$

$\therefore\ x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$ and 2π are critical points.

Moreover, $f(0) = \frac{7}{6}, f\left(\frac{2\pi}{3}\right) = \frac{-13}{12}, f(\pi) = \frac{-1}{6}, f(2\pi) = \frac{7}{6}, f\left(\frac{4\pi}{3}\right) = \frac{-13}{12}$.

$\therefore$ The required difference $= \frac{7}{6} - \left(\frac{-13}{12}\right) = \frac{9}{4}$ **Ans. :** (D)

(162) If $g(x) = \frac{1}{\log(1+x)} - \frac{1}{x}$, $x > 0$, then

(A) $1 < g(x) < 2$ (B) $-1 < g(x) < 0$ (C) $0 < g(x) < 1$ (D) $-\infty < g(x) < 0$

Sol. : For $x > 0$, let $f(x) = \log(1 + x)$.

Applying Mean Value theorem to $f(x)$ in the interval $[0, x]$, we get $c \in (0, x)$ such that,

$$\frac{\log(1+x) - \log(1+0)}{x - 0} = f'(c) = \frac{1}{1+c}$$

$$\therefore\ \frac{\log(1+x)}{x} = \frac{1}{1+c} < 1$$

$\log(1 + x) < x.$ So, $\frac{1}{\log(1+x)} - \frac{1}{x} > 0.$ Hence, $g(x) > 0$

Moreover, $c \in (0, x).$ So, $c < x$

$\therefore\ \frac{1}{1+c} > \frac{1}{1+x}.$ So, $\frac{\log(1+x)}{x} > \frac{1}{1+x}$

$\therefore \log(1+x) > \frac{x}{1+x}$

$\therefore \frac{1}{\log(1+x)} < \frac{1+x}{x}$. So, $g(x) = \frac{1}{\log(1+x)} - \frac{1}{x} < 1$. Hence, $g(x) < 1$ **Ans. :** (C)

(163) If the tangent at the point (1, 1) of the curve $y^2 = x(2-x)^2$ meets the curve again at the point P, then P is ..... .

(A) (4, 4) (B) (–1, 2) (C) $\left(\frac{9}{4}, \frac{3}{8}\right)$ (D) (–1, –1)

Sol. : $y^2 = x(2-x)^2$

$\therefore 2y\frac{dy}{dx} = (2-x)^2 + x \cdot 2(2-x)(-1) = 3x^2 - 8x + 4$

$\left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{1}{2}$

$\therefore$ The equation of the tangent at (1, 1) is $y - 1 = -\frac{1}{2}(x-1)$

$\therefore y = -\frac{1}{2}x + \frac{3}{2}$

Let us find the points of intersection of this tangent with the given curve.

$\left(-\frac{x}{2} + \frac{3}{2}\right)^2 = x(2-x)^2$

$\therefore x^2 - 6x + 9 = 16x + 4x^3 - 16x^2$

$\therefore 4x^3 - 17x^2 + 22x - 9 = 0$. So, $(x-1)(4x-9)(x-1) = 0$

$\therefore x = 1, \frac{9}{4}, 1$. So, $x = \frac{9}{4}$ and $y = \frac{3}{8}$. Hence, required point is $\left(\frac{9}{4}, \frac{3}{8}\right)$. **Ans. :** (C)

(164) The maximum volume of the cylinder inscribed in a right circular cone having height h and semi-vertical angle α is ..... .

(A) $4\pi h^3 \tan^2\alpha$ (B) $\frac{4}{27}\pi h^3 \tan^2\alpha$ (C) $\frac{1}{27}\pi h^3 \tan^2\alpha$ (D) $\frac{4}{27}\pi h^2 \tan^2\alpha$

Sol. : If the radius of the cylinder is x and semi vertical angle of the cone is α, then $\tan\alpha = \frac{x}{VN}$

$\therefore VN = x\cot\alpha$

$\therefore$ The height of the cylinder, $MN = VM - VN = h - x\cot\alpha$

$\therefore$ The volume of the cylinder, $f(x) = \pi x^2(h - x\cot\alpha)$

$= \pi x^2 h - \pi x^3 \cot\alpha$

$\therefore f'(x) = 0 \Rightarrow 2\pi hx - 3\pi x^2 \cot\alpha = 0$

$\therefore x = \frac{2}{3}h\tan\alpha$

$f''(x) = 2\pi h - 6\pi x \cot\alpha$

$\therefore f''\left(\frac{2}{3}h\tan\alpha\right) = -2\pi h < 0$

$\therefore$ The volume of the cylinder is maximum for $x = \frac{2}{3}h\tan\alpha$.

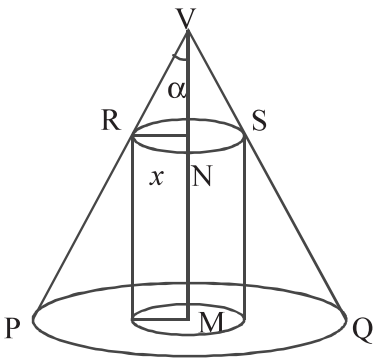


Figure 15.7

Maximum volume $f(x) = \pi x^2 MN = \pi \frac{4}{9} h^2 \tan^2 \alpha \left(h - \frac{2}{3}h\right)$

$= \frac{4\pi}{27} h^3 \tan^2 \alpha$ **Ans. :** (B)

(165) Which of the following lines passes through the point (1, 2) and forms a triangle of minimum area in the first quadrant ?

(A) $2x + y = 0$ (B) $2x + y = 2$ (C) $2x + y = 4$ (D) $x + 2y = 4$

Sol. : Let the given line be $\frac{x}{a} + \frac{y}{b} = 1$.

$\therefore$ The area of the triangle $A = \frac{1}{2} ab$

Since the line passes through (1, 2), we have $\frac{1}{a} + \frac{2}{b} = 1$. So, $b = \frac{2a}{a-1}$

$\therefore A = \frac{a^2}{a-1} = \frac{a^2 - 1 + 1}{a-1} = a + 1 + \frac{1}{a-1}$

$\therefore \frac{dA}{da} = 1 - \frac{1}{(a-1)^2} = 0 \Rightarrow a = 2$ or $a = 0$.

But $a = 0$ is not possible. So, $a = 2$

Moreover, $\frac{d^2A}{da^2} = \frac{2}{(a-1)^3}$. So, $\left(\frac{d^2A}{da^2}\right)_{a=2} = 2 > 0$

$\therefore$ The area A is minimum, when $a = 2$.

Now, if $a = 2$, then $b = \frac{2a}{a-1} = 4$. Hence, the required line is $\frac{x}{2} + \frac{y}{4} = 1$.

$\therefore 2x + y = 4$ **Ans. :** (C)

Note : From the options only one line $2x + y = 4$ passes through (1, 2). i.e. the required line.

(166) Let $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ be a polynomial in x where $0 < a_0 < a_1 ... < a_n$. $P(x)$ has,

(A) no maximum or minimum value (B) only one maximum value

(C) only one minimum value (D) n maximum values and n minimum values

Sol. : $P'(x) = 2a_1x + 4a_2x^3 + ... + 2na_nx^{2n-1}$

For maximum or minimum values, the necessary condition is $P'(x) = 0$.

$\therefore 2x (a_1 + 2a_2x^2 + ... + na_nx^{2n-2}) = 0$

$\therefore x = 0$ $(a_i > 0)$

Moreover, $P''(x) = 2a_1 + 12a_2x^2 + ... + 2n(2n-1)a_nx^{2n-2}$

$\therefore P''(0) = 2a_1 > 0$

$\therefore P(x)$ has a minimum value at $x = 0$.

Note : All terms of $P(x)$ other than constant term are non-negative. So, clearly a_0 is the only minimum value. **Ans. :** (C)

(167) The minimum value of $f(x) = |3 - x| + |2 + x| + |5 - x|$ is ..... .

(A) 0 (B) 7 (C) 8 (D) 10

Sol. : $f(x) = \begin{cases} 6-3x, & x < -2 \\ 10-x, & -2 \le x < 3 \\ x+4, & 3 \le x < 5 \\ 3x-6, & x \ge 5 \end{cases}$

$f'(x) = \begin{cases} -3, & x < -2 \\ -1, & -2 < x < 3 \\ 1, & 3 < x < 5 \\ 3, & x > 5 \end{cases}$

$\therefore$ f is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$. So, f has minimum value at $x = 3$ and minimum value is $f(3) = 7$. **Ans. :** (B)

(168) If $y(x) = cos\ (3\ cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm \frac{\sqrt{3}}{2}$, then $\frac{1}{y(x)} \left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ = ...

(A) 1 (B) 2 (C) 8 (D) 9

Sol. : $y = cos\ (3\ cos^{-1}x)$

$\frac{dy}{dx} = \frac{3\,sin(3cos^{-1}x)}{\sqrt{1-x^2}}$

$\therefore \sqrt{1-x^2}\ \frac{dy}{dx} = 3\ sin\ (3\ cos^{-1}x)$

$\frac{-x}{\sqrt{1-x^2}}\ \frac{dy}{dx} + \sqrt{1-x^2}\ \frac{d^2y}{dx^2} = 3\ cos\ (3\ cos^{-1}x)\ \left(\frac{-3}{\sqrt{1-x^2}}\right)$

$\therefore -x\ \frac{dy}{dx} + (1-x^2)\ \frac{d^2y}{dx^2} = -9y$

$\therefore \frac{1}{y}\left[(x^2-1)\ \frac{d^2y}{dx^2} + x\frac{dy}{dx}\right] = 9$ **Ans. :** (D)

: Questions Having More Than One Correct Answers :

(169) Let $f(x)$ be a polynomial of degree 3, where $f(0) + 2 = 0$ and $f'''(0) = 6$. If all three roots of $f(x) = 0$ are positive integers, then

(A) $f'(1) = 0$ (B) $2f'(n) + f''(0) = 0$ (C) $2f'(0) + f''(0) = 2$ (D) $f'(2) = 0$

Sol. : Let $f(x) = x^3 + ax^2 + bx - 2$ because $f'''(0) = 6$, $f(0) = -2$

Let $\alpha, \beta, \gamma \in N$ be the roots of $f(x) = 0$. So, $\alpha\beta\gamma = -\frac{-2}{1} = 2$

$\therefore$ Any two numbers out of α, β, γ would be 1 while the remaining number is 2.

Let $\alpha = 1, \beta = 1, \gamma = 2$,

$a = -(\alpha + \beta + \gamma) = -4 = \frac{f''(0)}{2}$ So, $f''(0) = -8$

$b = \alpha\beta + \beta\gamma + \alpha\gamma = 5 = f'(0)$. So, $f'(0) = 5$

$\therefore f(x) = x^3 - 4x^2 + 5x - 2$

$\therefore 2f'(0) + f''(0) = 2$

Also, 1 is a root of $f(x) = 0$ with multiplicity 2, so, $f'(1) = 0$ **Ans. :** (A), (C)

(170) If $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall x \in R$, then $g(x) = f(\sin x) + f(\cos x)$, $x \in \left[0, \frac{\pi}{2}\right]$ is

(A) decreasing in $\left(0, \frac{\pi}{4}\right)$

(B) increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(C) the function g has local minimum at $x = \frac{\pi}{4}$

(D) the function g has local maximum at $x = \frac{\pi}{4}$

Sol. : $g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

$g'(x) = 0 \quad \Rightarrow f'(\sin x)\cos x = f'(\cos x)\sin x$

$\Rightarrow f'(\sin x)\cos x = f'\left(\sin\left(\frac{\pi}{2} - x\right)\right)\cos\left(\frac{\pi}{2} - x\right)$

$\therefore x = \frac{\pi}{2} - x \qquad \left(x \in \left[0, \frac{\pi}{2}\right]\right) \qquad \left(f'(\sin x)\cos x \uparrow \text{ in } \left[0, \frac{\pi}{2}\right]\right)$

$\therefore x = \frac{\pi}{4}$

Now, $g''(x) = f''(\sin x)\cos^2 x + f''(\cos x)\sin^2 x - \{f'(\sin x)\sin x + f'(\cos x)\cos x\}$ (1)

Given, $f'(\sin x) < 0, \ \forall x \in R$

$\therefore f'(\cos x) < 0$

Using (1), $g''(x) > 0, \quad \forall x \in R$

$\therefore$ The function g has local minimum at $x = \frac{\pi}{4}$.

$\therefore g$ is decreasing in $\left(0, \frac{\pi}{4}\right)$ and increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. **Ans. :** (A), (B), (C)

(171) For which value of a, does $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have positive local maximum ?

(A) $a < -3$ (B) $-1 < a < 1$ (C) $3 < a < 4$ (D) $5 < a < 6$

Sol. : $f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

For $f(x)$ to be maximum $f'(x) = 0$

$\therefore 3x^2 + 6(a-7)x + 3(a^2-9) = 0$

$\therefore x^2 + 2(a-7)x + (a^2-9) = 0$

$\therefore x = -(a-7) \pm \sqrt{58-14a}$ (1)

x is real. So, $58 - 14a > 0$

$\therefore a < \frac{29}{7}$ (2)

Moreover, $f''(x) = 6x + 6(a-7) = 6(x + a - 7)$

If $x_1 = -(a-7) + \sqrt{58-14a}$, then $f''(x) = 6\sqrt{58-14a} > 0$

If $x_2 = -(a-7) - \sqrt{58-14a}$, then $f''(x) = -6\sqrt{58-14a} < 0$

$\therefore$ f has local maximum at $x_2 = -(a-7) - \sqrt{58-14a}$.

If this is positive then $-(a-7) - \sqrt{58-14a} > 0$

$\therefore$ $7 - a > \sqrt{58-14a}$. So, $a^2 - 9 > 0$

$\therefore$ $a < -3$ or $a > 3$ (3)

Using (2) and (3) we have, $a < -3$ or $3 < a < \frac{29}{7}$ i.e. $3 < a < 4$ **Ans. :** (A), (C)

(172) $\lim\limits_{n\to\infty} \frac{-3n+(-1)^n}{4n-(-1)^n} =$ .....

(A) $\frac{-3}{4}$ (B) $\frac{-3}{4}$ if n is even. (C) $\frac{-3}{4}$ is n is odd. (D) limit does not exist

Sol. : If $n \in$ N, then

$$\lim_{n\to\infty} \frac{-3n+(-1)^n}{4n-(-1)^n} = \lim_{n\to\infty} \frac{-3+\frac{(-1)^n}{n}}{4-\frac{(-1)^n}{n}} = \frac{-3}{4}$$ **Ans. :** (A), (B), (C)

(173) If $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x < 2 \\ 37-x, & 2 \le x \le 3 \end{cases}$ then

(A) f is increasing in (–1, 2) (B) f is continuous in [–1, 3].

(C) $f'(2)$ does not exist (D) f has local maximum at $x = 2$.

Sol. : f is polynomial. $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2+} f(x) = f(2) = 35$

$\therefore$ f is continuous on [–1, 3].

$$f'(x) = \begin{cases} 6x+12, & -1 < x < 2 \\ -1, & 2 < x < 3 \end{cases}$$

$f'(2-) = 24$, $f'(2+) = -1$. So, $f'(2)$ does not exist.

$f'(x) = 6x + 12 = 6(x+2) > 0$ $(-1 < x < 2)$

$\therefore$ f is an increasing function in (–1, 2).

Also, f is decreasing in (2, 3).

$\therefore$ f has a maximum at $x = 2$. **Ans. :** (A), (B), (C), (D)

(174) $f(x) = \lim\limits_{n\to\infty} \left\{ \frac{\cos \pi x - x^{2n} \sin(x-1)}{1+x^{2n+1}-x^{2n}} \right\}$

(A) f is continuous at $x = 1$. (B) $\lim\limits_{x\to 1} f(x) = 1$

(C) $\lim\limits_{x\to 1} f(x) = -1$ (D) $f(x) = \begin{cases} \cos \pi x & 0 < x < 1 \\ -1, & x = 1 \\ \frac{\sin(x-1)}{1-x}, & x > 1 \end{cases}$

Sol. : For $0 < x < 1$,

$$f(x) = \lim_{n\to\infty}\left\{\frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\} = \cos\pi x \qquad \left(\lim_{n\to\infty} x^n = 0\right)$$

For $x = 1$, $\quad f(x) = \lim_{n\to\infty}\left\{\frac{\cos\pi - \sin(1-1)}{1+1-1}\right\} = \cos\pi = -1$

For $x > 1$, $\quad f(x) = \lim_{n\to\infty}\left\{\frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}$

$$= \lim_{n\to\infty}\left\{\frac{\frac{\cos\pi x}{x^{2n+1}} - \frac{1}{x}\sin(x-1)}{\frac{1}{x^{2n+1}}+1-\frac{1}{x}}\right\} = \frac{\sin(x-1)}{(1-x)}$$

$$f(x) = \begin{cases} \cos\pi x & 0 < x < 1 \\ -1, & x = 1 \\ \dfrac{\sin(x-1)}{1-x}, & x > 1 \end{cases}$$

Now, $\lim_{x\to 1-} f(x) = \cos\pi x = -1$

$\lim_{x\to 1+} f(x) = -\lim_{x\to 1+}\frac{\sin(x-1)}{(x-1)} = -1$

and $f(1) = -1$. So, $\lim_{x\to 1} f(x) = f(1)$

$\therefore f$ is continuous at $x = 1$. **Ans. :** (A), (C), (D)

(175) If $f : R \to R$ is a polynomial function such that $f(2x) = f'(x)\,f''(x)$, then ..... (degree $n > 2$)

(A) $f(3) = 12$ (B) f is one-one but not onto

(C) $f(x) = x$ has three distinct real roots (D) f is one-one and onto

Sol. : Let, $f(x)$ be a polynomial of degree $n > 2$. $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ...+ a_0$

$\therefore f'$ is a polynomial of degree at most $n - 1$ and f'' is of degree at most $n - 2$.

Since $f(2x) = f'(x)\,f''(x)$, we have $n \geq n - 1 + n - 2$. But $n > 2$. So, $n = 3$.

$\therefore f$ is a polynomial of degree three.

Let $f(x) = ax^3 + bx^2 + cx + d$

Since $f(2x) = f'(x)\,f''(x)$, we have, $8ax^3 + 4bx^2 + 2cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

$\therefore 8ax^3 + 4bx^2 + 2cx + d = 18a^2x^3 + 18\,abx^2 + (4b^2 + 6ac)x + 2bc$

$\therefore 8a = 18a^2, \quad 4b = 18\,ab, \; 2c = 4b^2 + 6ac, \; d = 2bc$

$\therefore\ a = \frac{4}{9},\ b = 0,\ c = 0,\ d = 0$

$\therefore\ f(x) = \frac{4}{9}x^3$

$\therefore\ f(3) = 12,\ f$ is one-one and onto.

and if $f(x) = x$, then $\frac{4}{9}x^3 = x$

$\therefore\ x = 0, \pm\frac{3}{2}$

$f(x) = x$ has three distinct real solutions. **Ans. :** (A), (C), (D)

(176) If $P = \lim\limits_{y\to 0} \frac{1}{y}\int\limits_0^{\pi} tan(y\sin x)\,dx$ and $Q = \lim\limits_{n\to\infty} \int\limits_0^{n} \left(1-\frac{x}{n}\right)^n e^{\frac{x}{3}}\,dx$, then

(A) 3Q = 4P (B) 3P = 4Q (C) P + 2Q = 5 (D) 2(P + Q) = 7

Sol. : $P = \lim\limits_{y\to 0} \int\limits_0^{\pi} \sin x \frac{tan(y\sin x)}{y\sin x}\,dx = \int\limits_0^{\pi} \sin dx = 2$

$Q = \lim\limits_{n\to\infty} \int\limits_0^{n} \left(1-\frac{x}{n}\right)^n e^{\frac{x}{3}}\,dx$

$= \lim\limits_{n\to\infty} \int\limits_0^{n} e^{n\left[1-\frac{x}{n}-1\right]} e^{\frac{x}{3}}\,dx$

$= \lim\limits_{n\to\infty} \int\limits_0^{n} e^{-x} e^{\frac{x}{3}}\,dx$

$= \lim\limits_{n\to\infty} \int\limits_0^{n} e^{-\frac{2}{3}x}\,dx$

$= \lim\limits_{n\to\infty} \left[\frac{e^{-\frac{2}{3}x}}{-\frac{2}{3}}\right]_0^n = \frac{-3}{2}\lim\limits_{n\to\infty}\left[e^{-\frac{2}{3}n}-1\right] = \frac{3}{2}$

$\therefore\ P = 2,\ Q = \frac{3}{2}$

$\therefore\ 3P = 4Q,\ P + 2Q = 5,\ 2(P + Q) = 7$ **Ans. :** (B), (C), (D)

(177) If the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ are orthogonal to each other, then

(A) $\frac{1}{a}+\frac{1}{b} = \frac{1}{a'}+\frac{1}{b'}$ (B) $\frac{1}{a}+\frac{1}{b} = \frac{1}{a'}-\frac{1}{b'}$ (C) $\frac{1}{a}-\frac{1}{b} = \frac{1}{a'}+\frac{1}{b'}$ (D) $\frac{1}{a}-\frac{1}{b} = \frac{1}{a'}-\frac{1}{b'}$

Sol. : $ax^2 + by^2 = 1$

Differentiate with respect to x and we get, $2ax + 2by\frac{dy}{dx} = 0$

$\therefore\ m_1 = \dfrac{dy}{dx} = \dfrac{-ax}{by}$

Similarly for the curve $a'x^2 + b'y^2 = 1$ we have, $m_2 = \dfrac{-a'x}{b'y}$

If the given curves are orthogonal, then $m_1 m_2 = -1$.

$\therefore \left(\dfrac{-ax}{by}\right)\left(\dfrac{-a'x}{b'y}\right) = -1$

$\therefore aa'x^2 + bb'y^2 = 0$

Now, $ax^2 + by^2 = 1$

$a'x^2 + b'y^2 = 1$

$\therefore\ \dfrac{x^2}{b'-b} = -\dfrac{y^2}{a'-a} = \dfrac{1}{ab'-a'b}$

$aa'\ \dfrac{(b'-b)}{ab'-a'b} + bb'\ \dfrac{(a-a')}{ab'-a'b} = 0$

$\therefore\ \dfrac{b'-b}{bb'} + \dfrac{a-a'}{aa'} = 0 \qquad (ab' \neq a'b)$

$\therefore\ \dfrac{1}{b} - \dfrac{1}{b'} + \dfrac{1}{a'} - \dfrac{1}{a} = 0$

$\therefore\ \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a'} - \dfrac{1}{b'}$

Second method : $(a - a')x^2 + (b - b')y^2 = 0$

and $aa'x^2 + bb'y^2 = 0$

$\therefore -\dfrac{y^2}{x^2} = \dfrac{a-a'}{b-b'} = \dfrac{aa'}{bb'}$

$\therefore\ \dfrac{a-a'}{aa'} = \dfrac{b-b'}{bb'}$

$\therefore\ \dfrac{1}{a'} - \dfrac{1}{a} = \dfrac{1}{b'} - \dfrac{1}{b}$

$\therefore\ \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a'} - \dfrac{1}{b'}$

Ans. : (D)

(178) If $f(x) = \begin{cases} e^x, & 0 < x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, then

(A) g has a local minimum at $x = e$ and a local maximum at $x = 1 + \log 2$.

(B) g has a local maximum at $x = 1$ and a local minimum at $x = 2$.

(C) g has no maximum value.

(D) g has no minimum value.

Sol. : $g'(x) = \begin{cases} e^x, & 0 < x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$

For maximum or minimum value, $g'(x) = 0$.

For $0 < x < 1$, we have $e^x > 0$. Also let $2 - e^{x-1} = 0$ for some x such that, $1 < x < 2$.

$\therefore\ x = 1 + \log 2$

and for $2 < x < 3$, we have $x = e$ if $g'(x) = 0$

Now $g''(x) = \begin{cases} -e^{x-1}, & 1 < x < 2 \\ 1 \quad , & 2 < x < 3 \end{cases}$

$\therefore\ g''(1 + \log 2) = -e^{\log 2} = -2 < 0$

$\therefore\ g''(e) = 1 > 0$

$\therefore\ g$ has a local minimum at $x = e$ and a local maximum value at $x = 1 + \log 2$ **Ans. :** (A)

(179) If the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the line $\frac{x}{a} + \frac{y}{b} = 2$ at (a, b), then n has .....

(A) only 2 values (B) only 3 values

(C) only 4 values (D) any non-zero real number as value

Sol. : $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ Clearly (a, b) lies on the curve

Differentiating with respect to x,

$$\frac{n}{a}\left(\frac{x}{a}\right)^{n-1} + \frac{n}{b}\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx} = \frac{-\frac{n}{a}\left(\frac{x}{a}\right)^{n-1}}{\frac{n}{b}\left(\frac{y}{b}\right)^{n-1}}. \qquad \text{So, } \left(\frac{dy}{dx}\right)_{(a,b)} = -\frac{b}{a}$$

$\therefore$ The equation of the tangent at (a, b) is $y - b = -\frac{b}{a}(x - a)$.

$\therefore\ ay - ab = -bx + ab$

$\therefore\ bx + ay = 2ab$ So, $\frac{x}{a} + \frac{y}{b} = 2$

$\therefore$ For $n \in R - \{0\}$ the line $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve. **Ans. :** (D)

(180) Let a real function f be such that $f\left(\frac{x+y}{3}\right) = \frac{2f(x) + 2f(y) - 4}{6}$ $(x, y \in R)$. If $f'(0) = 2$, then f is

(A) increasing on R (B) decreasing on R

(C) increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$ (D) no decision about f

Sol. : Substituting $x = y = 0$ in the given equation.

$f(0) = \frac{4f(0) - 4}{6}$. So, $f(0) = -2$ (1)

Substituting $y = 0$ in the given equation.

$$\therefore\ f\left(\frac{x}{3}\right) = \frac{2f(x) + 2f(0) - 4}{6} = \frac{2f(x) - 4 - 4}{6} = \frac{f(x) - 4}{3}$$

$\therefore\ f\left(\frac{x}{3}\right) = \frac{f(x)-4}{3}$. So, $f(3x) = 3f(x) + 4$ (2)

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{2f(3x)+2f(3h)-4}{6}-f(x)}{h}$

$= \lim_{h\to 0} \frac{2f(3x)+2f(3h)-4-6f(x)}{6h}$

$= \lim_{h\to 0} \frac{6f(x)+8+2f(3h)-4-6f(x)}{6h}$ (From (2))

$= \lim_{h\to 0} \frac{f(3h)+2}{3h}$

$= \lim_{h\to 0} \frac{f(3h)-f(0)}{3h}$ (From (1))

$= f'(0) = 2$ $(f'(0) = 2)$

$\therefore\ f(x) = 2x + c$

$\therefore\ f$ is increasing in R. **Ans. :** (A)

(181) If the tangent at P (at^2, at^3) to the curve $ay^2 = x^3$ meets the curve again at Q(t'), then $t' =$ .....

(A) $2t$ (B) $-t$ (C) $\frac{t}{2}$ (D) $-\frac{t}{2}$

Sol. : $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3at^2}{2at} = \frac{3t}{2}$ (1)

The equation of the tangent at P (at^2, at^3) is $y - at^3 = \frac{3t}{2}(x - at^2)$

This tangent meets the curve again at $Q(t')$ = Q (at'^2, at'^3).

$\therefore$ From (1), $a(t'^3 - t^3) = \frac{3t}{2}\ a(t'^2 - t^2)$

$\therefore (t'^2 + t't + t^2)\ 2 = 3t\ (t'+t)$ $(t' \neq t)$

$\therefore 2t'^2 - t't - t^2 = 0$

$\therefore (t' - t)\ (2t' + t) = 0$

$\therefore t' = -\frac{t}{2}$ $(t' \neq t)$ **Ans. :** (D)

(182) If $y = a\log x + bx^2 + x$ has its extrema at $x = -1$ and $x = 2$, then

(A) $a = 2, b = -1$ (B) $a = 2, b = \frac{-1}{2}$ (C) $a = -2, b = \frac{1}{2}$ (D) None of these

Sol. : $y = a\log x + bx^2 + x$ has extrema at $x = -1$ and $x = 2$.

$\therefore \frac{dy}{dx} = 0$, at $x = -1$ and $x = 2$

$\therefore \frac{a}{x} + 2bx + 1 = 0$, at $x = -1$ and $x = 2$

$\therefore -a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0$

$\therefore a = 2$ and $b = \frac{-1}{2}$ **Ans. :** (B)

(183) For the curve $y = x^3$ the tangent at any point $P_1(P_1 \neq (0, 0))$ meets the curve again at P_2, the tangent at P_2 meets the curve again at P_3 and so on. X-coordinates of points $P_1, P_2, P_3, \ldots P_n$ are in .....

(A) arithmetic progression (B) geometric progression

(C) harmonic progression (D) arithmetico - geometric progression

Sol. : $y = x^3$

$\therefore \frac{dy}{dx} = 3x^2$. Let a point on the curve be $P_1 (h, h^3)$.

$\therefore \left(\frac{dy}{dx}\right)_{P_1} = 3h^2$

The equation of the tangent at P_1 is $y - h^3 = 3h^2 (x - h)$

$\therefore (x - h)(x^2 + xh + h^2) = 3h^2 (x - h)$ (For intersection with the curve.)

$\therefore x^2 + xh - 2h^2 = 0$ or $x = h$

$\therefore (x - h)(x + 2h) = 0$ $(x \neq h)$

$\therefore x = -2h$. So, $y = -8h^3$

$\therefore P_2 (-2h, -8h^3)$

$\therefore$ X-coordinate of P_3 is $-2(-2h) = 4h$ and y-coordinate $64h^3$

$\therefore P_3 = (4h, 64h^3)$

Proceeding similarly, we can see that X-coordinates of $P_1, P_2, P_3, \ldots P_n$ are $h, -2h, 4h, \ldots$ forming a geometric progression. **Ans. :** (B)

(184) If the tangent at $P\left(\frac{\pi}{4}\right)$ to the curve $x = \sec^2 \theta$, $y = \cot \theta$ intersects the curve again at Q, then PQ = ...

(A) $\sqrt{15}$ (B) $\frac{3}{2}\sqrt{5}$ (C) $\frac{1}{2}\sqrt{15}$ (D) $2\sqrt{15}$

Sol. : $\frac{dy}{dx} = \frac{-\text{cosec}^2 \theta}{2\sec^2\theta \tan\theta} = -\frac{1}{2}\cot^3 \theta$ So, $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -\frac{1}{2}$

$\therefore$ For $\theta = \frac{\pi}{4}$ we have $x = sec^2 \frac{\pi}{4} = 2$, $y = cot \frac{\pi}{4} = 1$

$\therefore$ The point on the curve $P\left(\frac{\pi}{4}\right)$ is P (2, 1).

The equation of the tangent at P is, $y - 1 = -\frac{1}{2}(x - 2)$

$\therefore$ $x + 2y - 4 = 0$ (1)

Now, $x = sec^2 \theta$, $y = \cot \theta$ for the curve

θ can be eliminated by using the formula $sec^2 \theta - tan^2 \theta = 1$. So, $x - \frac{1}{y^2} = 1$

The equation of the curve is $y^2 = \frac{1}{x-1}$ (2)

By (1), $y = \frac{4-x}{2}$

For the point of intersection, substitute $y = \frac{4-x}{2}$ in (2). So $\left(\frac{4-x}{2}\right)^2 = \frac{1}{x-1}$

$\therefore$ $(x^2 - 8x + 16)(x - 1) = 4$

$\therefore$ $x^3 - 9x^2 + 24x - 20 = 0$

$\therefore$ $(x - 2)(x^2 - 7x + 10) = 0$

$\therefore$ $(x - 2)^2 (x - 5) = 0$

$\therefore$ $x = 2$ or 5

If $x = 2$ then $y = 1$. If $x = 5$, then $y = -\frac{1}{2}$

$\therefore$ We get P(2, 1) (which is given) So, $Q = \left(5, -\frac{1}{2}\right)$

$\therefore$ PQ $= \sqrt{(5-2)^2 + \left(1+\frac{1}{2}\right)^2} = \frac{3\sqrt{5}}{2}$ **Ans. :** (B)

(185) If the tangent to the curve $y = ax^2 + bx + \frac{7}{2}$ at (1, 2) is parallel to the normal to the curve $y = x^2 + 6x + 10$ at (–2, 2), then a = ..., b = ...

(A) $-1, \frac{5}{2}$ (B) $\frac{5}{2}, -1$ (C) $1, -\frac{5}{2}$ (D) $-\frac{5}{2}, 1$

Sol. : $y = ax^2 + bx + \frac{7}{2}$ So, $\frac{dy}{dx} = 2ax + b$

So, $\left(\frac{dy}{dx}\right)_{(1,2)} = 2a + b$

Also, since the point (1, 2) lies on the curve, we have, $2 = a + b + \frac{7}{2}$. So, $a + b = -\frac{3}{2}$ (1)

Now, $y = x^2 + 6x + 10$

$\therefore$ $\frac{dy}{dx} = 2x + 6$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(-2,\ 2)} = 2$

So, Slope of normal $= -\frac{1}{2}$. By given condition, $2a + b = -\frac{1}{2}$ (2)

Solving (1) and (2) we have, $a = 1$, $b = -\frac{5}{2}$ **Ans. :** (C)

(186) If p_1 and p_2 are the lengths of the perpendiculars from origin to the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, then $4p_1^2 + p_2^2$ = ...

(A) $4a^2$ (B) $2a^2$ (C) a^2 (D) $3a^2$

Sol. : Parametric equations of given curve are $x = a\ sin^3\ \theta$, $y = a\ cos^3\ \theta$

$$\frac{dy}{dx} = \frac{-3acos^2\theta sin\theta}{3asin^2\theta cos\theta} = \frac{-cos\theta}{sin\theta}$$

The equation of the tangent is $y - acos^3\ \theta = \frac{-cos\theta}{sin\theta}(x - asin^3\ \theta)$

$\therefore\ x\ cos\theta + y\ sin\theta = asin\theta\ cos\theta$

The equation of the normal is $y - acos^3\ \theta = \frac{sin\theta}{cos\theta}\ (x - asin^3\ \theta)$

$\therefore\ x\ sin\theta - y\ sin\theta = -\ a(cos^4\theta - sin^4\theta)$

$\therefore\ x\ sin\theta - y\ \cos\theta = -\ acos2\theta$

$\therefore$ The equation of the tangent is $x\ cos\theta + y\ sin\theta = \frac{a}{2}\ sin\ 2\theta$

$$\therefore\ p_1 = \frac{\left|-\frac{a}{2}sin2\theta\right|}{\sqrt{cos^2\theta + sin^2\theta}} = \left|\frac{a}{2}sin2\theta\right|$$

$\therefore\ p_2 = |a\ cos\ 2\theta|$

$\therefore\ 4p_1^2 + p_2^2 = a^2$ **Ans. :** (C)

(187) $F(x) = \frac{x^3}{3} + \frac{3x^2}{2} + ax + b$. If $F(-2) = 0$, then how many ordered pairs (a, b) are possible for which F becomes minimum at P(–2, 0) ?

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $F(x) = \frac{x^3}{3} + \frac{3x^2}{2} + ax + b$

$F(-2) = 0 \Rightarrow 2a - b = \frac{10}{3}$

$F'(x) = x^2 + 3x + a$

$F'(-2) = 0 \Rightarrow a = 2$

$\therefore\ F'(x) = x^2 + 3x + 2 = (x + 1)\ (x + 2)$

$\therefore\ F'(x) = 0 \Rightarrow x = -1$ or -2

$\therefore\ F''(x) = 2x + 3$

For $x = -2$ we have $F''(x) = -1$

$\therefore$ We want F to have minimum value at $x = -2$. But $F''(-2) = -1$.

$\therefore$ No ordered pair (a, b) exists such that F has a minimum value at $(-2, 0)$. **Ans. :** (A)

: 'Match the Following' Type Questions :

(188)

Section 1	Section 2
(P) The area of the largest rectangle inscribed in an equilateral triangle of side 4.	(i) $2\sqrt{3}$
(Q) If the largest distance between points on curves $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 10x + 4y + 20 = 0$ is 3λ, then λ = ...	(ii) 0
(R) If $x^3 - 3x^2 - x + 3$, $x \in [0, 4]$ has maximum value $5k$, then k = ...	(iii) 9
(S) The minimum value of $cos^4 x - 6\ cos^2 x + 5$ is	(iv) 3

	(P)	(Q)	(R)	(S)
(A)	(iv)	(iii)	(ii)	(i)
(B)	(i)	(ii)	(iii)	(iv)
(C)	(i)	(iii)	(iv)	(ii)
(D)	(i)	(iii)	(ii)	(iv)

Sol. : (P) Let PQ = x

$\therefore$ BP = $\frac{4-x}{2}$

In Δ BPS, we have PS = $\frac{4-x}{2}$ $tan\ 60° = \frac{\sqrt{3}}{2}(4 - x)$

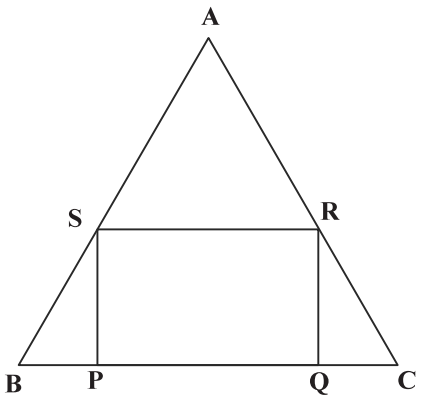


Figure 15.8

$\therefore$ The area of the rectangle PQRS

$\Delta = \frac{\sqrt{3}}{2}(4 - x)\ x$

$= \frac{\sqrt{3}}{2}(4x - x^2)$

$\therefore \frac{d\Delta}{dx} = \frac{\sqrt{3}}{2}(4 - 2x)$

$\therefore \frac{d\Delta}{dx} = 0 \Rightarrow x = 2.$ $\frac{d^2\Delta}{dx^2} = -\sqrt{3} < 0$

$\therefore \Delta$ is maximum at $x = 2$. Maximum $\Delta = \frac{\sqrt{3}}{2}(4 - 2)2 = 2\sqrt{3}$ (P) $\rightarrow$ (i)

(Q) $S_1 : x^2 + y^2 - 2x - 2y + 1 = 0$ is a circle. Centre C_1 (1, 1), radius $r_1 = 1$.

$S_2 : x^2 + y^2 - 10x + 4y + 20 = 0$ is a circle. Centre C_2 (5, –2), radius $r_2 = 3$

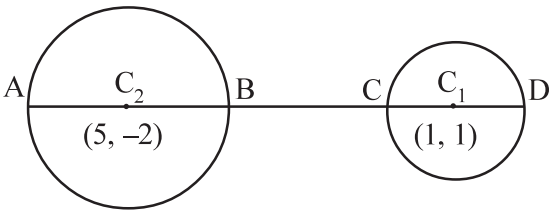


Figure 15.9

Maximum distance, AD = $C_1C_2 + r_1 + r_2 = 5 + 3 + 1 = 9$ (Q) → (iii)

(R) $y = x^3 - 3x^2 - x + 3$, $x \in [0, 4]$

$\frac{dy}{dx} = 0 \implies 3x^2 - 6x - 1 = 0 \implies x = \frac{6+4\sqrt{3}}{6}, \frac{6-4\sqrt{3}}{6}$

$y'' = 6x - 6 = -4\sqrt{3} < 0$, (taking $x = \frac{6-4\sqrt{3}}{6}$)

Now, global extreme value of y can be obtained at $x = 0$ or $x = 4$.

$f(0) = 3, f(4) = 15$, local maximum is smaller as $\frac{6-4\sqrt{3}}{6} < 0$

$\therefore$ The maximum value $15 = 5k \implies k = 3$ (R) → (iv)

(S) $y = \cos^4 x - 6\cos^2 x + 5$

Let $\cos^2 x = t$, then $0 \le t \le 1$

$\therefore y = t^2 - 6t + 5$, $0 \le t \le 1$ So, $\frac{dy}{dx} = 2t - 6 < 0$. y is decreasing function.

For $t \in [0, 1]$, y has minimum value at $t = 1$.

$y_{min} = 0$ (S) → (ii)

Shorter method : In (S) and (R) two options are the same

Solving (S), we get (S) → (ii).

(S) → (ii) in only (C). **Ans. :** (C)

(189) For the curve $y = \int_0^x 2|t|\,dt$, the equation of the tangent parallel to the bisector of the first quadrant .....

(A) $y = x \pm 1$ (B) $y = x \pm \frac{3}{4}$ (C) $y = x + \frac{3}{4}, y = x - \frac{1}{4}$ (D) $y = x \pm \frac{1}{2}$

Sol. : $y = \int_0^x 2|t|\,dt$

Differentiating with respect to x, $\frac{dy}{dx} = 2|x|$

Required tangent is parallel to the line which is a bisector of the first quadrant that is $y = x$. So, $\frac{dy}{dx} = 1$

$\therefore 2|x| = 1$. So, $x = \pm \frac{1}{2}$

$\therefore y = 2\int_0^{\frac{1}{2}} t\,dt = \frac{1}{4}$

$\therefore$ The point on the curve is $\left(\pm\frac{1}{2}, \frac{1}{4}\right)$.

$\therefore$ The equation of the tangent $y - \frac{1}{4} = 1\left(x - \left(\pm\frac{1}{2}\right)\right)$.

$\therefore$ $y = x + \frac{3}{4}$ and $y = x - \frac{1}{4}$ **Ans. :** (C)

(190) The tangent at any point to the ellipse $\frac{x^2}{8} + \frac{y^2}{18} = 1$ intersects coordinate axes at A and B. If the area of Δ OAB is minimum, then P is ..... .

(A) $(\sqrt{8}, 0)$ (B) $(0, \sqrt{18})$ (C) (2, 3) (D) $(1, \frac{\sqrt{63}}{2})$

Sol. : The equation of the tangent at any point P($a\cos\theta$, $b\sin\theta$) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

This tangent intersects axes at A $\left(\frac{a}{\cos\theta}, 0\right)$, B $\left(0, \frac{b}{\sin\theta}\right)$.

$\therefore$ The area of ΔOAB = $\frac{1}{2}\frac{|ab|}{\sin\theta\cos\theta} = \frac{|ab|}{\sin 2\theta}$

If this area is minimum, then $\sin 2\theta$ is maximum.

$\therefore \theta = \frac{\pi}{4}$

$\therefore$ P ($a\cos\theta$, $b\sin\theta$) = $\left(\frac{\sqrt{8}}{\sqrt{2}}, \frac{\sqrt{18}}{\sqrt{2}}\right) = (2, 3)$ ($a = \sqrt{8}$, $b = \sqrt{18}$) **Ans. :** (C)

(191) If the line $(3 - a)x + ay + a^2 = 1$ is a normal at some point of the curve $xy = 1$, then $a \in$.....

(A) (0, 3) (B) $(0, \infty)$ (C) (0, 1) (D) $(-\infty, 0) \cup (3, \infty)$

Sol. : $y = \frac{1}{x}$. So, $\frac{dy}{dx} = -\frac{1}{x^2}$

$\therefore$ The slope of the normal $x^2 > 0$

The line $(3 - a)x + ay + a^2 - 1 = 0$ is a normal.

$\therefore$ $-\frac{3-a}{a} > 0$. So, $\frac{a-3}{a} > 0$

$\therefore$ $a \in (-\infty, 0) \cup (3, \infty)$ **Ans. :** (D)

(192) The curve $y = ax^3 + bx^2 + cx + 5$ touches X-axis at a point P (–2, 0), Y-axis at a point Q and slope of tangent at Q is 3. Then a, b, c are ..... respectively.

(A) $-\frac{1}{2}, -\frac{3}{4}, 3$ (B) $3, -\frac{1}{2}, -\frac{3}{2}$ (C) $-\frac{3}{4}, -\frac{1}{2}, 3$ (D) $\frac{1}{2}, \frac{3}{2}, -3$

Sol. : The curve $y = ax^3 + bx^2 + cx + 5$ intersects Y-axis at Q (0, 5).

$\left(\frac{dy}{dx}\right)_{(0,5)} = 3$

$\therefore \left[3ax^2+2bx+c\right]_{x=0} = 3$ So, $c = 3$ (1)

Also, $\left(\frac{dy}{dx}\right)_{(-2,0)} = 0$ (The curve touches X-axis at the point P(–2, 0))

$\therefore$ $12a - 4b + c = 0$ So, $12a - 4b + 3 = 0$ (2)

Also, P(–2, 0) lies on the curve.

$\therefore$ $-8a + 4b - 2c + 5 = 0.$ So, $-8a + 4b - 1 = 0$ (3)

Solving (2) and (3) we get $a = -\frac{1}{2}$, $b = -\frac{3}{4}$ **Ans. :** (A)

(193) At which point of the curve in the first quadrant $ay^2 = x^3$, $a > 0$, the normal makes intercepts of equal length ?

(A) $\left(\frac{a}{9}, \frac{8a}{27}\right)$ (B) $\left(4a, \frac{8a}{27}\right)$ (C) $\left(\frac{4a}{9}, \frac{8a}{27}\right)$ (D) $\left(\frac{4a}{9}, \frac{-8a}{27}\right)$

Sol. : Let the requird point be (x_1, y_1).

$\therefore$ $ay_1^2 = x_1^3.$ So, $y_1 = \pm\sqrt{\frac{x_1^3}{a}}$

$\therefore$ Differentiate $ay^2 = x^3$ with respect to x

$2ay\ \frac{dy}{dx} = 3x^2$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1^2}{2ay_1} = \pm\frac{3x_1^2}{2a\sqrt{\frac{x_1^3}{a}}} = \pm\frac{3}{2}\sqrt{\frac{x_1}{a}}$

$\therefore$ The slope of the normal $= \mp\frac{2}{3}\sqrt{\frac{a}{x_1}}$

We know that the slope of a line which makes intercepts of equal length on axes $= \pm 1$.

$\therefore$ $\pm\frac{2}{3}\sqrt{\frac{a}{x_1}} = \pm 1.$ So, $\frac{4}{9}\ \frac{a}{x_1} = 1.$ So, $x_1 = \frac{4a}{9}$

$\therefore$ $y_1 = \pm\sqrt{\frac{64\ a^3}{9^3\ a}} = \pm\frac{8a}{27}$

$\therefore$ The required point in the first quadrant is $\left(\frac{4a}{9}, \frac{8a}{27}\right)$. **Ans. :** (C)

(194) Let a triangle be formed in the first quadrant by a tangent to the curve $f(x) = x^2 + bx - b$ at (1, 1) and coordinate axes. If the area of this triangle is 2 square units, then $b =$ .....

(A) –1 (B) 3 (C) –3 (D) 1

Sol. : $f(x) = x^2 + bx - b$ So, $f'(x) = 2x + b$

We have $f'(x) = 2 + b$ at (1, 1).

The equation of the tangent at (1, 1) is $y - 1 = (2 + b)(x - 1)$

This tangent intersects X-axis at A $\left(\frac{b+1}{b+2}, 0\right)$ and Y-axis at B(0, $-b-1$).

The area of Δ OAB = 2

$\therefore -\frac{1}{2}(b+1)\frac{(b+1)}{b+2} = 2$ So, $b^2 + 2b + 1 = -4b - 8$

$\therefore b^2 + 6b + 9 = 0$ Hence, $(b+3)^2 = 0$

$\therefore b = -3$ **Ans. :** (C)

(195) The equation of the tangent to the curve at $y = x^{\frac{1}{3}}(1 - \cos x)$ at $x = 0$ is ..... .

(A) $y = 0$ (B) $x = 0$ (C) $y = 1$ (D) $x = 1$

Sol. : $f'(x) = \frac{dy}{dx} = x^{\frac{1}{3}} \sin x + (1 - \cos x)\frac{1}{3}x^{-\frac{2}{3}}$

The function is continuous and differentiable at $x = 0$.

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^{\frac{1}{3}}(1-\cos h)}{h} = \lim_{h\to 0}\frac{h^{\frac{1}{3}}\sin\frac{h}{2}\sin\frac{h}{2}}{\frac{h}{2}} = 0$$

Also, since $x = 0$, $y = 0$

The equation of the tangent is $y = 0$. **Ans. :** (A)

(196) If the line joining points (0, 3) and (5, –2) is a tangent to the curve $y = \frac{c}{x+1}$, then c = ...

(A) 1 (B) –2 (C) 4 (D) 8

Sol. : The line joining (0, 3) and (5, –2) is $y - 3 = \frac{3+2}{0-5}(x - 0) = -x$

$\therefore x + y - 3 = 0$ is a tangent to the curve at some point.

For point of contact (α, β), $\alpha + \beta - 3 = 0$ (1)

Now, $y = \frac{c}{x+1}$. So, $\frac{dy}{dx} = \frac{-c}{(x+1)^2}$

$\therefore \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{-c}{(\alpha+1)^2} = -1$ (The slope of the line $x + y - 3 = 0$ is –1) (2)

$\therefore c = (\alpha + 1)^2$ and $\beta = \frac{c}{\alpha+1}$ ((α, β) lies on the curve)

$\therefore -\frac{c}{\left(\frac{c}{\beta}\right)^2} = -1$

$\therefore \beta^2 = c$ or $(3 - \alpha)^2 = c = (\alpha + 1)^2$

$\therefore 3 - \alpha = \pm(\alpha + 1)$

$\therefore 3 - \alpha = \alpha + 1$ So, $\alpha = 1$

$\therefore c = (1 + 1)^2 = 4$ **Ans. :** (C)

(197) Which of the following statement/s is/are true for the function $f(x) = \frac{\tan^2 x - \cot^2 x + 1}{\tan^2 x + \cot^2 x - 1}$?

(A) $f(x)$ has a local maximum value $\frac{5}{3}$. (B) $f(x)$ has a local minimum value -1

(C) $f(x)$ does not have global maximum value. (D) $f(x)$ does not have global minimum value

Sol. : $f(x) = \frac{\tan^2 x - \cot^2 x + 1}{\tan^2 x + \cot^2 x - 1} = \frac{\tan^4 x + \tan^2 x - 1}{\tan^4 x - \tan^2 x + 1}$

Let $\tan x = t$. $g(t) = \frac{t^4 + t^2 - 1}{t^4 - t^2 + 1}$

$g'(t) = \frac{-4t^5 + 8t^3}{\left(t^4 - t^2 + 1\right)^2} = \frac{4t^3(2 - t^2)}{\left(t^4 - t^2 + 1\right)^2}$

For maximum or minimum, $g'(t) = 0$

$\therefore\ t^2 = 2$ ($t = 0$ is not possible.)

In the neighbourhood of $t^2 = 2$, $g'(t)$ changes sign from positive to negative.

$\therefore$ $g(t)$ has local maximum value when $t^2 = 2$

$g(\sqrt{2}) = \frac{5}{3}$

$\therefore$ The function f has a local maximum value as $\frac{5}{3}$.

Also, f has no minimum value. **Ans. :** (A), (D)

(198) $f(x) = ax^3 + bx^2 + cx + d$ is an increasing function where $a, b, c, d \in R$ and $3b^2 < c^2$.

Let $g(x) = af'(x) + bf''(x) + c^2$. If $G(x) = \int_{\alpha}^{x} g(t)\,dt$, $\alpha \in R$, then for $\alpha < x < \alpha + 1$

(A) $G(x)$ is a decreasing function (B) $G(x)$ is an increasing function

(C) $G(x)$ is neither increasing nor decreasing (D) $G(x)$ is a one-one function

Sol. : $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c > 0$ ($f(x)$ an increasing function)

$\therefore\ a > 0$ and $4b^2 - 12ac < 0$

$\therefore\ a > 0$ and $b^2 < 3ac$

Moreover, $g(x) = af'(x) + bf''(x) + c^2$

$\therefore\ g(x) = a(3ax^2 + 2bx + c) + b(6ax + 2b) + c^2$

$= 3a^2x^2 + 8abx + 2b^2 + c^2 + ac$

$\Delta = 64a^2b^2 - 4(3a^2)(2b^2 + c^2 + ac)$

$= 4a^2(10b^2 - 3c^2 - 3ac) < 4a^2(10b^2 - 3c^2 - b^2)$ ($b^2 < 3ac$)

$\therefore\ \Delta < 4a^2(9b^2 - 3c^2) = 12a^2(3b^2 - c^2) < 0$ ($3b^2 < c^2$)

$\therefore\ g(x) > 0, \forall\ x \in R$

$\therefore\ G(x) = \int_{\alpha}^{x} g(t)\,dt$ is an increasing function.

$\therefore\ G(x)$ is a one-one function. **Ans. :** (B), (D)

(199) The tangent at P to the curve $y = x\ tan\ \alpha - \frac{1}{2}\ \frac{x^2}{u^2\ \cos^2\ \alpha}$, $\alpha \in \left(0, \frac{\pi}{2}\right)$ is parallel to the line $y = x + 5$. If y-coordinate of P is $\frac{u^2}{4}$, then α = ...

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$

Sol. : $y = x\ tan\ \alpha - \frac{1}{2}\ \frac{x^2}{u^2\ \cos^2\ \alpha}$

$\therefore \frac{dy}{dx} = tan\ \alpha - \frac{x}{u^2\ \cos^2\ \alpha}$

The tangent is parallel to the line $y = x + 5$. So, $\frac{dy}{dx} = 1$

$tan\ \alpha - \frac{x}{u^2\ \cos^2\ \alpha} = 1$

$\therefore\ x = (tan\ \alpha - 1)\ u^2\ cos^2\ \alpha$, and $y = \frac{u^2}{4}$ (given)

Substituting in $y = x\ tan\ \alpha\ -\frac{1}{2}\frac{x^2}{u^2\ cos^2\ \alpha}$,

$\therefore\ \frac{u^2}{4} = (tan\ \alpha - 1)\ u^2\ cos^2\ \alpha\ \left[tan\ \alpha - \frac{1}{2}(tan\ \alpha - 1)\right]$

$\therefore\ \frac{1}{4} = (tan\ \alpha - 1)\ cos^2\ \alpha\ \left[\frac{tan\alpha + 1}{2}\right]$

$\therefore\ \frac{1}{2} = cos^2\ \alpha\ (tan^2\ \alpha - 1) = sin^2\ \alpha\ - cos^2\ \alpha = -\ cos\ 2\alpha$

$\therefore\ cos\ 2\alpha = -\frac{1}{2}$. So, $2\alpha = \frac{2\pi}{3}$ $\left(\alpha \in \left(0, \frac{\pi}{2}\right)\right)$

$\therefore\ \alpha = \frac{\pi}{3}$ **Ans. :** (D)

(200) The height of the cylinder inscribed in a sphere of radius R and having maximum volume is ..... .

(A) $\frac{2R}{\sqrt{3}}$ (B) $\frac{R}{\sqrt{2}}$ (C) $\frac{5R}{4}$ (D) R

Sol. : If the radius of the cylinder is r and the height h, then

$\left(\frac{h}{2}\right)^2 + r^2 = R^2$ (R = the radius of the sphere)

$\therefore \frac{h^2}{4} + r^2 = R^2$

$r^2 = R^2 - \frac{h^2}{4}$

The volume of cylinder $V = \pi r^2 h$

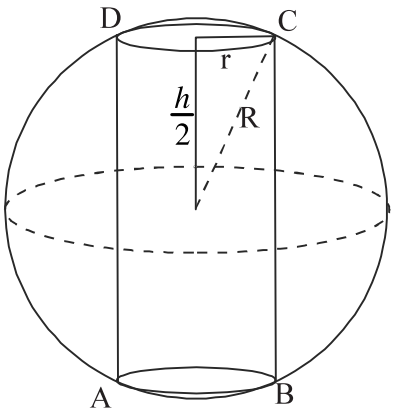


Figure 15.10

$$= \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi\left(R^2h - \frac{h^3}{4}\right)$$

$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right) = 0 \Rightarrow \pi\left(R^2 - \frac{3h^2}{4}\right) = 0.$ So, $R^2 = \frac{3}{4}h^2.$ So, $h = \frac{2}{\sqrt{3}}$ R

$\frac{d^2V}{dh^2} = \pi\left(0 - \frac{6h}{4}\right) = -\frac{3}{2}\pi h.$ Since $h = \frac{2}{\sqrt{3}}$ R we have $\frac{d^2V}{dh^2} < 0$

$\therefore$ When $h = \frac{2}{\sqrt{3}}$ R, the volume of the cylinder is maximum. **Ans. :** (A)

(201) How many solutions does the equation $x^3 + 2x^2 + 5x + 2\ cos\ x = 0$ have in $[0, 2\pi]$?

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $f(x) = x^3 + 2x^2 + 5x + 2\ cos\ x$

$\therefore f'(x) = 3x^2 + 4x + 5 - 2\ sin\ x$

$= 3\left(x + \frac{2}{3}\right)^2 + \frac{11}{3} - 2\ sin\ x$

Now, $\frac{11}{3} - 2\ sin\ x > 0, \forall\ x$ $(-1 \le sin\ x \le 1)$

$\therefore f'(x) > 0, \forall\ x$ So, f is an increasing function.

Moreover, $f(0) = 2.$ Also, $f(x) > f(0)$

$\therefore f(x) > 2$

$\therefore f(x) = 0$ has no solution in $[0, 2\pi]$. **Ans. :** (A)

(202) The minimum value of the function $f(x) = x^{\frac{3}{2}} + x^{\frac{-3}{2}} - 4\left(x + \frac{1}{x}\right)$ is ..... .

(A) 0 (B) 10 (C) –10 (D) does not exist

Sol. : $f(x) = x^{\frac{3}{2}} + \frac{1}{x^{\frac{3}{2}}} - 4\left(x + \frac{1}{x}\right)$

Substituting $t = \sqrt{x}$ we get $f(t) = t^3 + \frac{1}{t^3} - 4\left(t^2 + \frac{1}{t^2}\right)$ $(t > 0)$

$= \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right) - 4\left[\left(t + \frac{1}{t}\right)^2 - 2\right]$

$\therefore\ f(t) = \left(t + \frac{1}{t}\right)^3 - 4\left(t + \frac{1}{t}\right)^2 - 3\left(t + \frac{1}{t}\right) + 8$

Substituting $t + \frac{1}{t} = a,$

$f(a) = a^3 - 4a^2 - 3a + 8,$ where $a \ge 2$ $(t > 0)$

$f'(a) = 3a^2 - 8a - 3 = (3a + 1)(a - 3)$

$f'(a) = 0 \Rightarrow a = -\frac{1}{3}$ or 3

But $a \ge 2.$ So, $a = 3$

In $[2, 3)$, $f'(a) < 0$ and in $(3, \infty)$, $f'(a) > 0$

The minimum value $f(3) = 3^3 - 4 \cdot 3^2 - 3 \cdot 3 + 8 = -10$ **Ans. :** (C)

(203) If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ has a local maximum at $x = x_1$ and a local minimum at $x = x_2$ where $x_2 = x_1^2$, then a = ...

(A) 0 (B) $\frac{1}{4}$ (C) 2 (D) (A) or (C)

Sol. : $f'(x) = 6x^2 - 18ax + 12a^2$

For the maximum or minimum values, $f'(x) = 0$

$\therefore x^2 - 3ax + 2a^2 = 0$

$\therefore (x - a)(x - 2a) = 0$

$\therefore x = a$ or $2a$

Now, $x_2 = x_1^2$, so $a < 0$ is not possible. So, $a \geq 0$

If $a = 0$, then $f'(x) = 6x^2 \geq 0$

$\therefore f$ is increasing function which is not possible. So, $a > 0$

$f''(x) = 12x - 18a$

$= 6(2x - 3a)$

$\therefore$ The function f has a local minimum at $x = 2a$ and a local maximum at $x = a$.

$\therefore x_1 = a$ and $x_2 = 2a$

$\therefore 2a = a^2 \Rightarrow a = 2$ $(a \neq 0)$ **Ans. :** (C)

(204) If a tangent to the curve $y = f(x)$ at the point whose x-coordinate is 1 makes an angle of measure $\frac{\pi}{6}$, makes an angle of measure $\frac{\pi}{3}$ at the point whose x-coordinate is 2, and makes an angle of measure $\frac{\pi}{4}$ at the point whose x-coordinate is 3 with positive direction of X-axis, then $\int_1^3 f'(x) f''(x)\,dx + \int_2^3 f''(x)\,dx$ = ...

(A) $\frac{3\sqrt{3}}{4}$ (B) $\frac{4+\sqrt{3}}{3}$ (C) $\frac{4+3\sqrt{3}}{3}$ (D) $\frac{4-3\sqrt{3}}{3}$

Sol. : We have $\frac{dy}{dx} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ at $x = 1$. So, $f'(1) = \frac{1}{\sqrt{3}}$. Similarly, $f'(2) = \sqrt{3}$, $f'(3) = 1$

$$\therefore \int_1^3 f'(x)\, f''(x)\,dx + \int_2^3 f''(x)\,dx = \int_{f'(1)}^{f'(3)} t\,dt + [f'(x)]_2^3 \quad \text{(taking } f'(x) = t, \text{ we have } f''(x)dx = dt)$$

$$= \frac{1}{2}\left[t^2\right]_{f'(1)}^{f'(3)} + f'(3) - f'(2)$$

$$= \frac{1}{2}\left[[f'(3)]^2 - [f'(1)]^2\right] + 1 - \sqrt{3}$$

$$= \frac{1}{2}\left[1 - \frac{1}{3}\right] + 1 - \sqrt{3} = \frac{4-3\sqrt{3}}{3}$$ **Ans. :** (D)

(205) If the tangent to the curve $x^3 - y^2 = 0$ at the point P $(4m^2, 8m^3)$ is a normal to the curve at some other point Q, then $9m^2$ = ...

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $y^2 = x^3$ (1)

$2y\frac{dy}{dx} = 3x^2$. So, $\frac{dy}{dx} = \frac{3x^2}{2y}$

$\therefore \left(\frac{dy}{dx}\right)_{x=4m^2} = \frac{3\cdot 16m^4}{2\cdot 8m^3} = 3m$

The equation of the tangent at P is $y - 8m^3 = 3m\,(x - 4m^2)$

$\therefore\ y = 3mx - 4m^3$ (2)

This tanget meets the curve at some other point Q.

$\therefore$ By using (1) and (2), $(x - m^2)(x - 4m^2)^2 = 0$. So, $x = 4m^2$ or m^2

If $x = 4m^2$, then $y = 8m^3$ which are coordinates of P.

$\therefore\ x = m^2$ and $y = -m^3$ (From (2)). So, Q is $(m^2, -m^3)$

$\therefore$ The slope of the tangent at Q is

$\left(\frac{dy}{dx}\right)_{x=m^2} = -\frac{3}{2}m$. So, slope of the normal $= \frac{2}{3m}$

It tangent at P is normal at Q, then

$\frac{2}{3m} = 3m$ Hence, $9m^2 = 2$ **Ans. :** (B)

(206) The measure of the angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ is .....

(A) $tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ (B) $tan^{-1}\left(\frac{5\sqrt{2}}{7}\right)$ (C) $tan^{-1}\left(\frac{3\sqrt{2}}{7}\right)$ (D) $tan^{-1}\left(\frac{2\sqrt{2}}{7}\right)$

Sol. : Given curve $y = \begin{cases} x^2 - 1, & |x| \geq 1 \\ 1 - x^2 & |x| < 1 \end{cases}$ and (1)

$y = \begin{cases} x^2 - 3, & |x| \geq \sqrt{3} \\ 3 - x^2 & |x| < \sqrt{3} \end{cases}$ (2)

First we find points of intersection of the two curves.

$|x^2 - 1| = |x^2 - 3|$

$\therefore\ x^2 - 1 = \pm\,(x^2 - 3)$

$\therefore\ x = \pm\sqrt{2}$ and $y = 1$

$\therefore$ The points of intersection are A $(\sqrt{2}, 1)$ and B $(-\sqrt{2}, 1)$.

We find the measure of the angle at A$(\sqrt{2}, 1)$.

By using (1) we have, $\frac{dy}{dx} = 2x$ So, $\left(\frac{dy}{dx}\right)_A = 2\sqrt{2} = m_1$. Similarly, $m_2 = -2\sqrt{2}$

$tan\,\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{4\sqrt{2}}{7}$

$\therefore\ \theta = tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

At point B also, we get the same measure. **Ans. :** (A)

(207) Let $f'(x)$ exist for the function f, for all $x \in R$. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$, $x \in R$. Then

(A) If f is increasing, then h is also increasing.

(B) If f is increasing, then h is decreasing.

(C) If f is decreasing, then h is increasing.

(D) Nothing can be said about h.

Sol. : $h'(x) = f'(x) - 2f(x)\,f'(x) + 3\,f(x)^2\,f'(x)$

$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$

$= 3f'(x)\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$

If $f'(x) > 0$, then $h'(x) > 0$

If f is increasing then h is also increasing. **Ans. :** (A)

(208) If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are functions such that $min\ f(x) > max\ g(x)$, then what is the relation between b and c ?

(A) no relation (B) $0 < c < \frac{b}{2}$ (C) $|c| < |b|\sqrt{2}$ (D) $|c| > |b|\sqrt{2}$

Sol. : Here, $f(x) = x^2 + 2bx + 2c^2 = (x + b)^2 + 2c^2 - b^2$

$\therefore min\ f(x) = 2c^2 - b^2$

and $g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2$

$\therefore max\ g(x) = b^2 + c^2$

Now, $min\ f(x) > max\ g(x)$

$\therefore\ 2c^2 - b^2 > b^2 + c^2$ So, $c^2 > 2b^2$ So, $|c| > \sqrt{2}\,|b|$ **Ans. :** (D)

(209) The angle between the curves $y = [\,|sin\ x| + |cos\ x|]$ and $x^2 + y^2 = 10$, has measure ..... where $[x]$ is an integer part function.

(A) $tan^{-1}\ 3$ (B) $tan^{-1}\ (-3)$ (C) $tan^{-1}\ \sqrt{3}$ (D) $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Sol. : $|sin\ x| + |cos\ x| = \sqrt{(|sin x| + |cos x|)^2}$

$= \sqrt{1 + |sin\ 2x|}$

Now, $0 \le |sin\ 2x| \le 1$

$\therefore\ 1 \le \sqrt{1 + |sin\ 2x|} \le \sqrt{2}$

$\therefore\ [|sin\ x|] + |cos\ x|] = 1$

$\therefore\ y = 1$ (1) and $x^2 + y^2 = 10$ (2)

Solving (1) and (2) we get points of intersection as (3, 1), (–3, 1).

Now, $x^2 + y^2 = 10$

$\therefore\ \frac{dy}{dx} = -\frac{x}{y}$

$\therefore\ \left(\frac{dy}{dx}\right)_{(3,1)} = -3$ and $\left(\frac{dy}{dx}\right)_{(-3,1)} = 3$

Also, for $y = 1$, $\frac{dy}{dx} = 0 = m_2$

$\therefore$ The angle between the curves at (3, 1) has measure

$tan^{-1}\left|\frac{-3-0}{1+(-3)0}\right| = tan^{-1}\ 3$

The angle between curves at (–3, 1) has also measure $tan^{-1}\ 3$. **Ans. :** (A)

(210) A tangent is drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at the point $(3\sqrt{3}\ cos\ \theta,\ sin\ \theta)$. For which value of θ, is the sum of the intercepts made by this tangent minimum ? $\left(\theta \in \left(0, \frac{\pi}{2}\right)\right)$

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{4}$

Sol. : The equation of the tangent to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\ cos\ \theta,\ sin\ \theta)$ is

$\frac{x\cos\theta}{3\sqrt{3}} + y\ sin\ \theta = 1$ $\left(\theta \in \left(0, \frac{\pi}{2}\right)\right)$

The sum of the intercepts $s = 3\sqrt{3}\ sec\ \theta + cosec\ \theta$

$\therefore\ \frac{ds}{d\theta} = 3\sqrt{3}\ sec\ \theta\ tan\ \theta - cosec\ \theta\ cot\ \theta$

For s to be minimum $\frac{ds}{d\theta} = 0$. So, $\frac{3\sqrt{3}\ sin\theta}{cos^2\theta} - \frac{cos\theta}{sin^2\theta} = 0$

$\therefore\ \frac{1}{3\sqrt{3}} = tan^3\ \theta$

$\therefore\ tan\ \theta = \frac{1}{\sqrt{3}}$ So, $\theta = \frac{\pi}{6}$ $\left(\theta \in \left(0, \frac{\pi}{2}\right)\right)$

Moreover, $\frac{d^2s}{d\theta^2} = 3\sqrt{3}\ sec\ \theta\ tan^2\ \theta + 3\sqrt{3}\ sec^3\ \theta + cosec\theta\ cot^2\ \theta + cosec^3\ \theta$

$\therefore\ \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} > 0$

$\therefore$ s is minimum at $\theta = \frac{\pi}{6}$. **Ans. :** (B)

(211) Let $f(x) = (1 + x)^n - (1 + nx)$, $x \in [-1, \infty)$. f has .....

(A) global maximum at $x = 0$.

(B) no global maximum or global minimum at $x = 0$.

(C) global minimum at $x = 0$

(D) we can't say anything about maximum or minimum at $x = 0$

Sol. : $f(x) = (1 + x)^n - (1 + nx)$

$\therefore\ f'(x) = n(1 + x)^{n-1} - n.$ So, $f'(0) = 0$

Also for $x > 0$, $f'(x) > 0$

$\therefore$ f is increasing on $[0, \infty)$.

$\therefore$ $f(x) \geq f(0) = 0$ $\forall\ x \in [0, \infty)$

Now, for $-1 \le x < 0$, $0 \le 1 + x < 1$

$\therefore (1 + x)^{n-1} - 1 < 0$

$\therefore f$ is decreasing on $[-1, 0]$.

$\therefore f(x) \ge f(0) = 0 \qquad \forall\, x \in [-1, 0)$

$\therefore f(x) \ge f(0) = 0 \qquad \forall\, x \in [-1, \infty)$

$\therefore f$ has global minimum at $x = 0$. **Ans. :** (C)

(212) A tangent and normal drawn to the curve $y^2 = 4ax$ at the point P$(at^2, 2at)$ intersect X-axis at T and N respectively. The angle between the tangents drawn at the point P of parabola and the tangent drawn at the point P of the circle passing through T, P, N has measure .....

(A) $tan^{-1}\, t^2$ (B) $cot^{-1}\, t^2$ (C) $tan^{-1}\, |t|$ (D) $cot^{-1}\, |t|$

Sol. : The equation of the tangent to the curve $y^2 = 4ax$ at the point (x_1, y_1) is $yy_1 = 2a\,(x + x_1)$.

$\therefore$ The equation of the tangent at P $(at^2, 2at)$ is

$ty = x + at^2$ (1)

The slope of the tangent $= \dfrac{1}{t}$

This tangent intersects the X-axis at T $(-at^2, 0)$. (2)

The equation of the normal at this point is $y = -tx + 2at + at^3$

It intersects X-axis at N $(2a + at^2, 0)$. The circle passing through the points T, P, N has end points of diameter as T and N because $\overleftrightarrow{TP}$ and $\overleftrightarrow{NP}$ are perpendicular to each other.

$\therefore$ The equation of the circle is $(x + at^2)\,(x - 2a - at^2) + y^2 = 0$.

$\therefore x^2 + y^2 - 2ax - at^2\,(2a + at^2) = 0$

Differentiating with respect to x we get,

$2x + 2y\,\dfrac{dy}{dx} = 2a \qquad$ So, $\dfrac{dy}{dx} = \dfrac{a - x}{y}$

$\left(\dfrac{dy}{dx}\right)_{(at^2, 2at)} = \dfrac{a - at^2}{2at} = \dfrac{1 - t^2}{2t}$ which is the slope of the tangent at P for circle. (3)

From (2) and (3),

$$tan\,\theta = \left|\frac{\frac{1}{t} - \frac{1-t^2}{2t}}{1 + \frac{1-t^2}{2t^2}}\right| = \left|\frac{2t - t + t^3}{2t^2 + 1 - t^2}\right| = \left|\frac{t + t^3}{t^2 + 1}\right| = |t|$$

$\therefore \theta = tan^{-1}\, |t|$

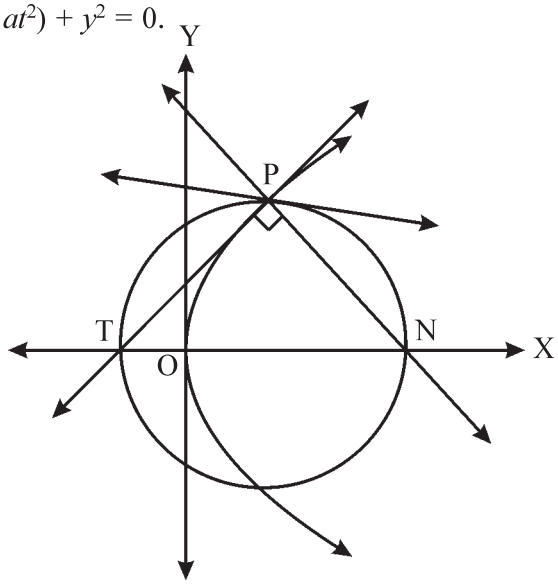


Figure 15.11

Ans. : (C)

(213) Let $f(x) = \begin{cases} x^3 + x^2 - 10x, & -1 \le x < 0 \\ \sin x, & 0 \le x < \dfrac{\pi}{2} \\ 1 + \cos x, & \dfrac{\pi}{2} \le x < \pi \end{cases}$, then f

(A) has a local maximum at $x = \dfrac{\pi}{2}$ (B) has a local minimum at $x = \dfrac{\pi}{2}$

(C) has global minimum at $x = 0$ (D) has global maximum at $x = \dfrac{\pi}{2}$

Sol. : $f'(x) = \begin{cases} 3x^2+2x-10, & -1<x<0 \\ \cos x, & 0<x<\frac{\pi}{2} \\ -\sin x, & \frac{\pi}{2}<x<\pi \end{cases}$

The function f is not differentiable at $x = 0$ and $x = \frac{\pi}{2}$.

because $f'(0–) = –10, f'(0+) = 1, f'\left(\frac{\pi}{2}-\right) = 0, f'\left(\frac{\pi}{2}+\right) = -1$

$\therefore$ Critical points of f are $x = 0, \frac{\pi}{2}$.

For $0 < x < \frac{\pi}{2}$, we have $f'(x) > 0$ and for $\frac{\pi}{2} < x < \pi$, $f'(x) < 0$.

$\therefore$ f has local maximum at $x = \frac{\pi}{2}$.

Also, for $-1 < x < 0$, we have $f'(x) < 0$ $(-12 < f'(x) < -7)$ and for $0 < x < \frac{\pi}{2}$, we have $f'(x) > 0$.

$\therefore$ f has local minimum at $x = 0$. Also f has global minimum at $x = 0$. **Ans. :** (A), (C)

(214) If the triangle formed by a tangent at any point of the curve $x^n y = a^n$ and coordinate axes has constant area, then n = ...

(A) $\frac{1}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) 1

Sol. : $x^n y = a^n$

$nx^{n-1} y + x^n \frac{dy}{dx} = 0$ So, $\frac{dy}{dx} = \frac{-n x^{n-1} y}{x^n} = \frac{-n y}{x}$

Let (x_0, y_0) be any point on the curve. So, $\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = -\frac{n y_0}{x_0}$

The equation of the tangent at this point is $y - y_0 = -\frac{n y_0}{x_0}(x - x_0)$

If this tangent intersects X-axis and Y-axis at A and B respectively, then

A $\left(x_0\left(\frac{1+n}{n}\right), 0\right)$ and B $(0, (1 + n) y_0)$. O is the origin.

Area of Δ OAB $= \frac{1}{2} \frac{(1+n)}{n} x_0 (1 + n) y_0$

$= \frac{1}{2} \frac{(1+n)^2}{n} x_0 \frac{a^n}{x_0^n}$ $((x_0, y_0)$ lies on the curve$)$

$= \frac{1}{2} a^n \frac{(1+n)^2}{n} \frac{1}{x_0^{n-1}}$

If this area is constant, then $x_0^{n-1} = 1$

$\therefore$ $n - 1 = 0$. So, $n = 1$ **Ans. :** (D)

(215) The point at which the function $y = 1 + a^2x - x^3$ is minimum with x satisfying the inequality $\frac{x^2+x+2}{x^2+5x+6} \le 0$, then the parameter $a \in$ ..... .

(A) $\emptyset$ (B) $(-3\sqrt{3}, -2\sqrt{3})$ (C) $(2\sqrt{3}, 3\sqrt{3})$ (D) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

Sol. : $y = 1 + a^2x - x^3$

$\therefore \frac{dy}{dx} = a^2 - 3x^2$. Let $\frac{dy}{dx} = 0$ So, $a^2 - 3x^2 = 0$. Hence $x = \pm\frac{a}{\sqrt{3}}$

$\frac{d^2y}{dx^2} = -6x$. So, y is minimum at $x = -\frac{a}{\sqrt{3}}$

Now, for any $x \in$ R, $x^2 + x + 2 > 0$

$\therefore \frac{x^2+x+2}{x^2+5x+6} \le 0 \Leftrightarrow x^2 + 5x + 6 < 0$

$\therefore$ If $x = -\frac{a}{\sqrt{3}}$, we should have, $\left(-\frac{a}{\sqrt{3}}\right)^2 - 5\frac{a}{\sqrt{3}} + 6 < 0$

$\therefore a^2 - 5\sqrt{3}\,a + 18 < 0$

$\therefore (a - 2\sqrt{3})(a - 3\sqrt{3}) < 0$

$\therefore a \in (2\sqrt{3}, 3\sqrt{3})$ **Ans. :** (C)

(216) For which value of a is $f(x) = -\left(\frac{b^2-ab-2}{b^2-3b+4}+1\right)x^5 - 3x + \log 5$, $b \in$ R a decreasing function on R ?

(A) $(-1, 2)$ (B) $(0, 7)$ (C) $[-7, 1]$ (D) $(2, 7)$

Sol. : $f(x) = -\left(\frac{b^2-ab-2}{b^2-3b+4}+1\right)x^5 - 3x + \log 5$

$f'(x) = -\left(\frac{b^2-ab-2}{b^2-3b+4}+1\right)5x^4 - 3$

If f is a decreasing function on R, then $f'(x) \le 0$, $\forall x \in$ R

$\therefore -\left(\frac{b^2-ab-2}{b^2-3b+4}+1\right) \le 0$

$\therefore \left(\frac{b^2-ab-2}{b^2-3b+4}+1\right) \ge 0$

$\therefore \frac{2b^2-b(3+a)+2}{b^2-3b+4} \ge 0$

$\therefore\ b^2 - 3b + 4 > 0$ (as $\Delta < 0,\ a > 0$) and $2b^2 - b\ (3 + a) + 2 \geq 0$

$\Rightarrow (3 + a)^2 - 4\ (2)\ (2) \leq 0$ $\qquad (\Delta \leq 0)$

$\Rightarrow 9 + a^2 + 6a - 16 \leq 0$

$\Rightarrow a^2 + 6a - 7 \leq 0$

$\Rightarrow (a + 7)\ (a - 1) \leq 0$

$\Rightarrow a \in [-7, 1]$ **Ans. :** (C)

(217) Composite function $f_1\ (f_2\ (f_3\ (...(f_n\ (x)))))$ is a decreasing function. Among given n functions r functions are decreasing functions and remaining functions are increasing functions. Maximum value of $r(n-r)$ is ..... .

(A) $\frac{n^2-4}{4}$, $n = 4k,\ k \in N$ (B) $\frac{n^2}{4}$, n even number

(C) $\frac{n^2-1}{4}$, n odd number (D) $\frac{n^2}{4}$, $n = 4k + 2,\ k \in N$

Sol. : Let $g(x) = f_1\ (f_2\ (f_3\ (...(f_n\ (x)))))$

$g(x)$ is a decreasing function. So, $g'(x) < 0$

Now, $g'(x) = f'_1\ (f_2\ (f_3\ (...(f_n\ (x))))) \times f'_2\ (f_3\ (...(f_n\ (x)))....) \times (f'_{n-1}\ (f_n(x)) \times f'_n\ (x)$

Since $g'(x) < 0$, among n terms above an odd number of terms must have negative values.

r functions are given to be decreasing.

$\therefore\ r$ is odd. We have to find maximum value of $r\ (n - r)$.

Let $P(r) = rn - r^2,\ 0 < r \leq n$

$\therefore\ P'(r) = n - 2r$, and $P''(r) = -2$

$\therefore\ P(r)$ is maximum when $r = \frac{n}{2}$.

We note that r is odd. Let us we think of different cases.

Case 1 : (a) $n = 4m + 2,\ m \in N$

In this case $r = \frac{n}{2}$ is odd.

$\therefore\ P\ (r) = \frac{n^2}{4}$

Case 1 : (b) $n = 4m,\ m \in N$

In this case $\frac{n}{2}$ is not odd.

$\therefore$ We think values before $\frac{n}{2}$ and after $\frac{n}{2}$.

$r = \frac{n}{2} - 1$ or $r = \frac{n}{2} + 1$

In both cases $P\ (r) = \frac{n^2-4}{4}$

Now, $(r + n - r)^2 + [r - (n - r)]^2 = 4r(n - r)$

$r(n - r)$ is maximum because $r + n - r = n$ is given.

For minimum, $r - (n - r) = 2r - n = 0, \pm 1, \pm 2, ...$

$r = \frac{n}{2}$ is not odd.

$r = \frac{n+1}{2}$ is not integer.

$r = \frac{n \pm 2}{2}$. So, $P(r) = \frac{n^2-4}{4}$

Case 2 : n is odd.

$r(n-r)$ is maximum if $2r-n$ is minimum.

r is odd.

$\therefore \quad 2r-n = \pm 1$

$\therefore \quad r = \frac{n+1}{2}$

$\therefore \quad P(r) = r(n-r) = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right) = \frac{n^2-1}{4}$ **Ans. :** (A), (C), (D)

(218) If a point (u, v) on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ is farthest from the point $(0, -2)$, then $u + v =$ .....

(A) 2 (B) 3 (C) 5 (D) 8

Sol. : Here, $4x^2 + a^2y^2 = 4a^2$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{4} = 3$

Let $(u, v) = (a\ cos\ \theta, 2\ sin\ \theta)$ be a point on the curve. The square of the distance from point $(0, -2)$ is

$f(\theta) = a^2\ cos^2\ \theta + (2\ sin\ \theta + 2)^2$

$f'(\theta) = 0 \Rightarrow 2a^2 cos\theta(-\ sin\ \theta) + 8(sin\ \theta + 1)\ cos\theta = 0$

$\therefore \ cos\ \theta\ \{(8 - 2a^2)\ sin\ \theta + 8\} = 0$

$\therefore \quad \theta = \frac{\pi}{2}$ or $sin\,\theta = \frac{4}{a^2-4}$

Now, $4 < a^2 < 8$

$\therefore 0 < a^2 - 4 < 4.$ So, $0 < \frac{a^2-4}{4} < 1$

$\therefore \frac{4}{a^2-4} > 1.$ But $sin\,\theta \le 1$ So, $\theta = \frac{\pi}{2}$

$\therefore$ The point on the curve is $(0, 2)$ So, $u + v = 0 + 2 = 2$ **Ans. :** (A)

(219) Let $f : [0, 1]$ be twice differentiable function. Also, $f(0) = f(1) = 0$ and $f''(x) - 2f'(x) + f(x) \ge e^x$. If the function $e^{-x} f(x)$ has minimum value in the interval $[0, 1]$ at $x = \frac{1}{4}$, then which of the following statement is true ?

(A) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$ (B) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$

(C) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$ (D) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$

Sol. : Let, $g(x) = e^{-x} f(x)$

$\therefore \ g'(x) = e^{-x}(f'(x) - f(x))$

$\therefore \ g''(x) = e^{-x}(f''(x) - 2f'(x) + f(x))$

Since $(f''(x) - 2f'(x) + f(x)) \ge e^x$, $x \in [0, 1]$

$\therefore \ e^{-x}(f''(x) - 2f'(x) + f(x)) \ge 1$

$\therefore \ g''(x) \ge 1 > 0$

$\therefore \ g'(x)$ is increasing.

$0 < x < \frac{1}{4} \Rightarrow g'(x) < g'\left(\frac{1}{4}\right)$ (g has minimum at $x = \frac{1}{4}$)

$\Rightarrow e^{-x}(f'(x) - f(x)) < 0$

$\Rightarrow f'(x) < f(x)$

$\therefore f'(x) < f(x), \quad 0 < x < \frac{1}{4}$ **Ans. :** (C)

(220) For which value of the x function $f(x) = 2|x| + |x + 2| - \big||x + 2| - 2|x|\big|$ has local maximum or local minimum ?

(A) –2 (B) $-\frac{2}{3}$ C) 2 (D) $\frac{2}{3}$

Sol. : $y = 2|x| + |x + 2| - \big||x+2| - 2|x|\big|$

Case 1 : $x < -2$. So, $x < 2$. So, $x - 2 < 0$

$\therefore y = -2x - x - 2 - |-x - 2 + 2x| = -2x - x - 2 - |x - 2| = -3x - 2 - 2 + x$

$\therefore y = -2x - 4$ (1)

Case 2 : $y = -2x + x + 2 - |x + 2 + 2x|$ $-2 < x < 0$

$y = -x + 2 - |3x + 2|$

If $-2 < x < -\frac{2}{3}$, then $y = 2x + 4$ (2)

and If $-\frac{2}{3} \le x < 0$, then $y = -4x$ (3)

Case 3 : $y = 2x + x + 2 - |x + 2 - 2x|$ $x \ge 0$

$y = 3x + 2 - |x - 2|$

If $0 \le x < 2$, then $y = 4x$ (4)

If $x \ge 2$, then $y = 2x + 4$ (5)

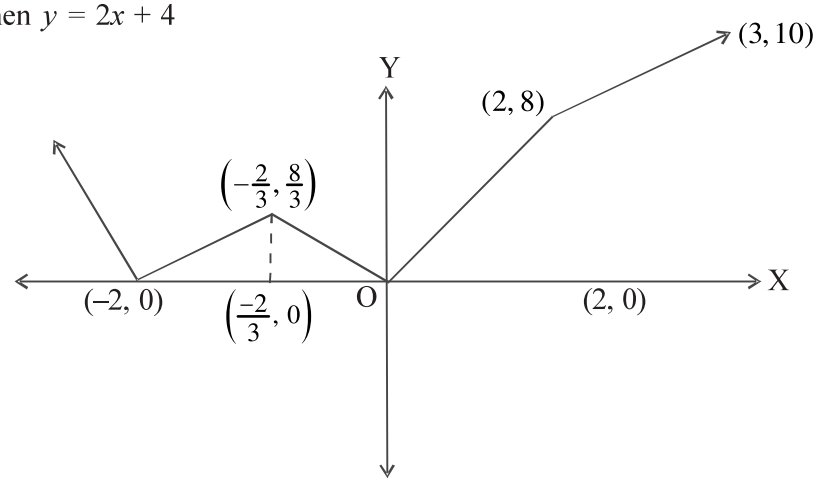


Figure 15.12

From the graph we see that f has a local minimum at $x = -2, 0$ and a local maximum at $x = -\frac{2}{3}$.

Ans. : (A), (B)

(221) For the curve $y^2 = 8x$ and $xy = -1$, the equation of the common tangent is ..... .

(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$

Sol. : Any point of the curve $xy = -1$ is of the form $\left(t, -\frac{1}{t}\right)$.

Now, $y = -\frac{1}{x}$. So, $\frac{dy}{dx} = \frac{1}{x^2}$

$\therefore \left(\frac{dy}{dx}\right)_{(t,-\frac{1}{t})} = \frac{1}{t^2}$

$\therefore$ For the curve $xy = -1$, the equation of the tangent at the point $\left(t, -\frac{1}{t}\right)$ is

$y + \frac{1}{t} = \frac{1}{t^2}(x - t)$ i.e. $y = \frac{1}{t^2}x - \frac{2}{t}$ (1)

If this line is a tangent to the curve $y^2 = 8x$, then it is of the form, $y = mx + \frac{2}{m}$. ($a = 2$)

$\therefore$ Since $m = \frac{1}{t^2}$, $\frac{2}{m} = -\frac{2}{t}$. So, $m = -t$. Hence, $-t = \frac{1}{t^2} \Rightarrow t = -1$

$\therefore$ By using (1) the equation of common tangent to both the curves is $y = x + 2$ **Ans. :** (D)

(222) The minimum value of $y = \frac{b^2}{a-x} + \frac{a^2}{x}$, $0 < x < a$, $a, b > 0$ is ..... .

(A) $\frac{a+b}{a}$ (B) $\frac{ab}{a+b}$ (C) $\frac{1}{a} + \frac{1}{b}$ (D) $\frac{(a+b)^2}{a}$

Sol. : $\frac{dy}{dx} = \frac{-b^2}{(a-x)^2}(-1) - \frac{a^2}{x^2}$.

Taking $\frac{dy}{dx} = 0$, we get, $bx = a^2 - ax$ or $a^2 - ax = -bx$ ($0 < x < a$, $a,b > 0$)

$\therefore x = \frac{a^2}{a+b}$ ($0 < x < a$) So, $a - x = \frac{ab}{a+b}$

$\frac{d^2y}{dx^2} = \frac{2a^2}{x^3} + \frac{2b^2}{(a-x)^3}$ So, $\left(\frac{d^2y}{dx^2}\right)_{x = \frac{a^2}{a+b}} > 0$

$\therefore y$ is minimum at $x = \frac{a^2}{a+b}$.

$\therefore y_{min} = \frac{b^2(a+b)}{ab} + \frac{a^2(a+b)}{a^2} = (a + b)\left[\frac{b}{a}+1\right] = \frac{(a+b)^2}{a}$ **Ans. :** (D)

(223) Let P(x) be a polynomial of degree four and $\lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right) = 2$. If P($x$) has maximum or minimum at $x = 1, 2$, then the value of P(2) = ...

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $\lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right) = 2$. So, $\lim_{x\to 0}\frac{P(x)}{x^2} = 1$

Let $P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$

Since $\lim_{x\to 0}\frac{P(x)}{x^2} = 1$. So, $a_3 = a_4 = 0$ and $a_2 = 1$

$\therefore P(x) = a_0x^4 + a_1x^3 + x^2$

Also, P(x) has maximum or minimum at $x = 1, 2$. So, $P'(1) = P'(2) = 0$

$P'(x) = 4a_0x^3 + 3a_1x^2 + 2x$

$\therefore\ 4a_0 + 3a_1 = -2$ and $32a_0 + 12a_1 = -4$

Solving these two equations, we get, $a_0 = \frac{1}{4}$, $a_1 = -1$

$\therefore\ P(x) = \frac{1}{4}x^4 - x^3 + x^2$. So, $P(2) = \frac{1}{4}\ 16 - 8 + 4 = 0$ **Ans. :** (A)

(224) If $g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$ and we have $f''(x) < 0$ for $0 \le x \le 1$, then $g(x)$ is

(A) decreasing in $\left[0, \frac{2}{3}\right)$ (B) decreasing in $\left(\frac{2}{3}, 1\right)$

(C) increasing in $\left(0, \frac{2}{3}\right)$ (D) increasing in $\left(\frac{2}{3}, 1\right]$

Sol. : $f''(x) < 0$. So, $f'(x)$ is a decreasing function.

Now, $g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$

$\therefore\ g'(x) = f'\left(\frac{x}{2}\right) - f'(1-x)$

If $g(x)$ is increasing function, then $g'(x) > 0$

$\therefore\ f'\left(\frac{x}{2}\right) > f'(1-x)$. So, $\frac{x}{2} < 1 - x$. So, $x < \frac{2}{3}$ **Ans. :** (B), (C)

(225) For the function $f : R \longrightarrow R$, we have $f(x)f(y) - f(xy) = x + y$ and $f(1) > 0$. Also, $h(x) = f(x)\, f^{-1}(x)$. The length of largest interval in which $h\ (sinx + cosx)$ is strictly inceasing is ..... .

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) π

Sol. : $f(x)f(y) - f(xy) = x + y,\ \forall\ x, y \in R$

Taking $x = 1, y = 1$, $(f(1))^2 - f(1) - 2 = 0$ (1)

$\therefore\ (f(1) - 2)\ (f(1) + 1) = 0$

Also, $f(1) > 0$. So, $f(1) = 2$

Substituting, $y = 1$ in (1) we get, $f(x) \cdot 2 - f(x) = x + 1$

$\therefore\ f(x) = x + 1$ So, $f^{-1}(x) = x - 1$

$\therefore\ h(x) = f(x)\ f^{-1}(x) = x^2 - 1$

$h(sin\ x + cos\ x) = (sin\ x + cos\ x)^2 - 1 = sin\ 2x$

We know that the length of largest interval in which $sinx$ is strictly increasing is π.

$\therefore$ For $sin\ 2x$ it is $\frac{\pi}{2}$. **Ans. :** (C)

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

(A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.

(B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.

(C) Statement 1 is true, statement 2 is false.

(D) Statement 1 is false, statement 2 is true.

(226) **Statement 1 :** $f(x) = \begin{cases} 2x+2 & , -4 \le x < 1 \\ 4x^2 - 16x + 16, & 1 \le x \le 4 \end{cases}$ has no maximum or minimum value.

Statement 2 : Even if a function is not differentiable at some point, then also the function may have maximum or minimum value at that point.

Sol. : $f(x) = \begin{cases} 2(x+1); & -4 \le x < 1 \\ 4(x-2)^2; & 1 \le x \le 4 \end{cases}$

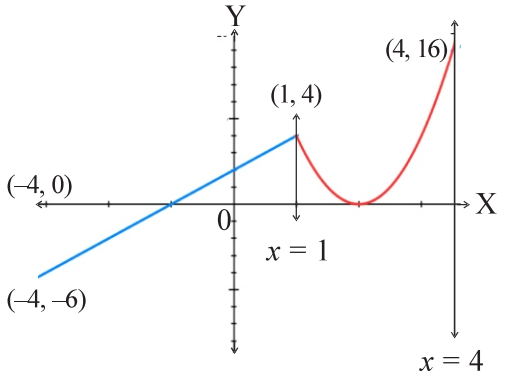


Figure 15.13

From the figure it is clearly seen that $f(x)$ has local maximum at $x = 1$ but it has global maximum at $x = 4$ which is 16.

Statement 1 : is not true.

Statement 2 : is true. f is not differentiable at $x = 1$.

Ans. : (D)

(227) **Statement 1 :** One root of the equation $x^3 - 2x^2 - 1 = 0$ lies between 2 and 3.

Statement 2 : If for a continuous function f, $f(a)$ and $f(b)$ are of opposite sign, then there is at least one root of $f(x) = 0$ between a and b.

Sol. : **Statement 1 :** $f(x) = x^3 - 2x^2 - 1$, $f(2) = -1 < 0, f(3) = 8 > 0$

$\therefore$ $f(2)$ and $f(3)$ are of opposite signs.

Also, $f(x)$ is continuous.

$\therefore$ f has at least one root between 2 and 3.

Statement 1 is true.

Statement 2 is true.

Statement 2 gives the correct explanation of statement 1.

Ans. : (A)

(228) $f(x) = \begin{cases} 2^x + 1, & -1 \le x < 0 \\ 2^x, & x = 0 \\ 2^x - 1, & 0 < x \le 1 \end{cases}$

Statement 1 : Function f is bounded but it has no maximum or minimum.

Statement 2 : f is discountinuous at $x = 0$.

Sol. : In the interval $[-1, 0)$, the value of function f increases from $\frac{3}{2}$ to 2 and on $(0, 1]$ the value of function increases from 0 to 1.

The function f does not take the value 0 or 2.

The function f is bounded but does not have maximum or minimum.

This happens because function f is discontinuous at $x = 0$.

Statement 1 and statement 2 are true. Statement 1 does not depend on statement 2. **Ans. :** (B)

(229) Let $f(x) = tan^{-1} x$ and $g(x) = x - \frac{x^3}{6}$

Statement 1 : $f(x) < g(x)$ $\qquad (0 < x < 1)$

Statement 2 : $h(x) = tan^{-1} x - x + \frac{x^3}{6}$ is a decreasing function on $[0, 1]$.

Sol. : Let $h(x) = tan^{-1} x - x + \frac{x^3}{6}$

$\therefore \quad h'(x) = \frac{1}{1+x^2} - 1 + \frac{x^2}{2} = \frac{2-2(1+x^2)+(1+x^2)x^2}{2(1+x^2)} = \frac{x^2(x^2-1)}{2(1+x^2)}$

If $x \in [0, 1]$, then $x^2 - 1 \leq 0$

$\therefore \quad h'(x) \leq 0$

$\therefore$ h is decreasing in $[0, 1]$. Statement 2 is true.

Also, $0 < x < 1$

$\therefore \quad h(0) > h(x).$ $\qquad$ So, $h(x) < 0$

$\therefore \quad tan^{-1} x - x + \frac{x^3}{6} < 0.$ $\qquad$ So, $tan^{-1} x < x - \frac{x^3}{6}$

$\therefore \quad f(x) < g(x)$

$\therefore$ Statement 1 is true. Statement 2 gives the correct explanation of 1. **Ans. :** (A)

(230) **Statement 1 :** $e^{\pi} > \pi^e$

Statement 2 : Function $\frac{1}{x^x}$ $(x > 0)$ has local maximum at $x = e$.

Sol. : Let $f(x) = \frac{1}{x^x}$ $\qquad$ So, $\log f(x) = \frac{\log x}{x}$

$\therefore \quad \frac{f'(x)}{f(x)} = \frac{x.\frac{1}{x} - \log x}{x^2} = \frac{1-\log x}{x^2}$

$\therefore \quad f'(x) = \frac{1}{x^x}\left(\frac{1-\log x}{x^2}\right)$

If $x > e$, then $f'(x) < 0$ and if $0 < x < e$, then $f'(x) > 0$

$\therefore$ f has local maximum at $x = e$.

$\therefore$ Statement 2 is true.

Also, $\pi > e$ and f is a decreasing function on (e, ∞).

$\therefore$ $f(\pi) < f(e)$ i.e. $\pi^{\frac{1}{\pi}} < e^{\frac{1}{e}}$ i.e. $\pi^e < e^\pi$

$\therefore$ Statement 1 is true.

Statement 2 does not give correct explanation of statement 1. **Ans. :** (B)

(231) **Statement 1 :** $|\tan x - \tan y| < |x - y|, \forall\ x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Statement 2 : If a function f is differentiable in some open interval and $|f'(x)| \leq M$, then $|f(x) - f(y)| \leq M\,|x - y|$

Sol. : Applying mean value theorem on $[x, y]$ to f, there exists some $c \in (x, y)$ such that

$\frac{f(y) - f(x)}{y - x} = f'(c)$

$|f(y) - f(x)| = |f'(c)|\,|y - x| \leq M\,|y - x|$ $\qquad (|f'(x)| \leq M)$

Statement 2 is true.

If $|f'(x)| \geq M$, then $|f(y) - f(x)| \geq M\,|y - x|$

$\therefore$ If $f(x) = \tan x$, we have $f'(x) = \sec^2 x$. So, $|f'(x)| \geq 1$

$\therefore$ $|\tan x - \tan y| \geq |y - x|$

$\therefore$ Statement 1 is not true. **Ans. :** (D)

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