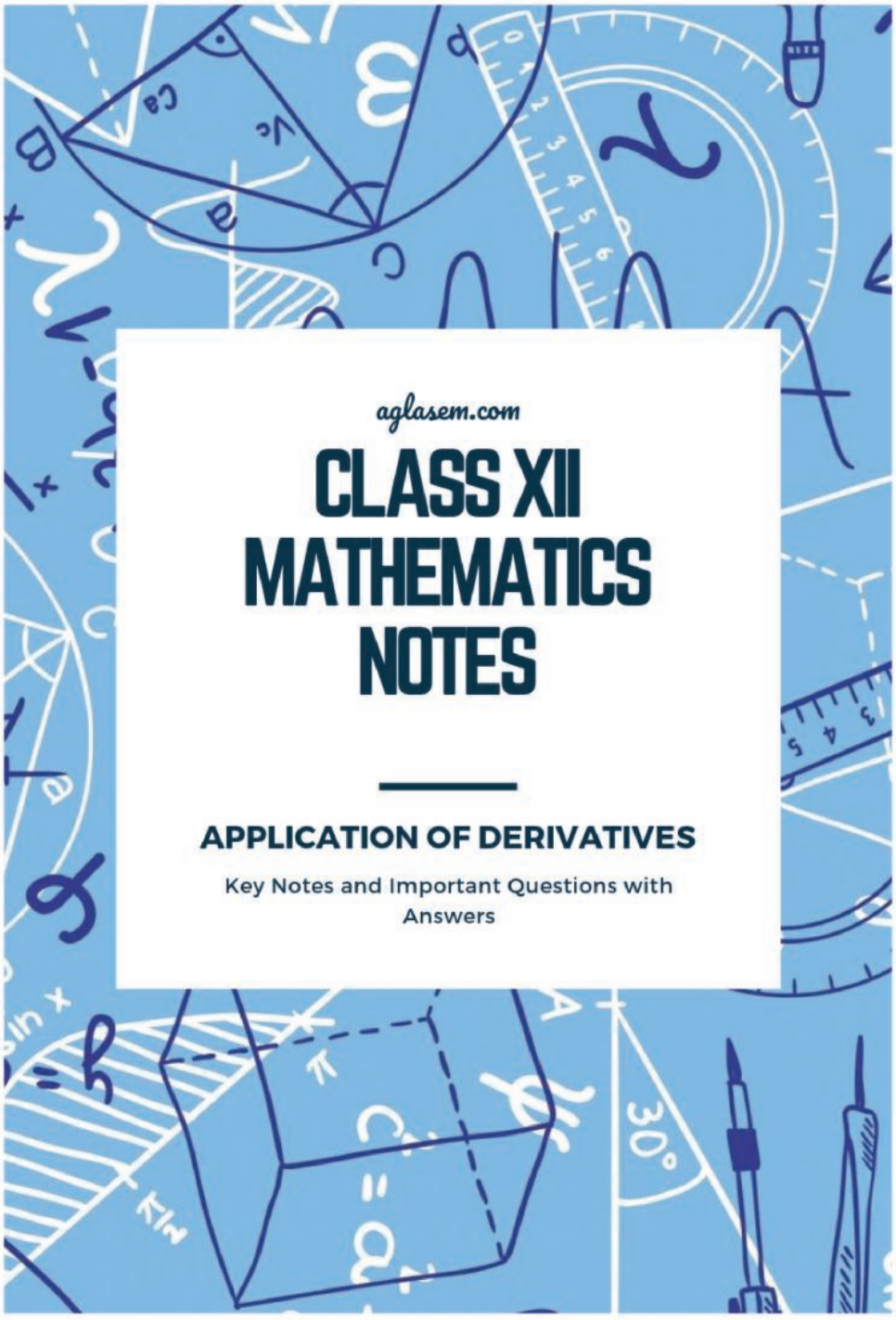


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CLASS XII MATHEMATICS NOTES

APPLICATION OF DERIVATIVES

Key Notes and Important Questions with Answers

CHAPTER 6

APPLICATION OF DERIVATIVES

IMPORTANT POINTS TO REMEMBER

- **Rate of change:** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\left\{\frac{dy}{dx}\right\}_{x = x_1}$ or $f'(x_1)$represents the rate of change of y w.r.t. x at $x = x_1$.

- **Increasing and Decreasing Function**

  Let f be a real-valued function and let I be any interval in the domain of f. Then f is said to be

  a) Strictly increasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

  b) Increasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

  c) Strictly decreasing in I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

  d) Decreasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

- **Derivative Test:** Let f be a continuous function on [a, b] and differentiable on (a, b). Then

a) f is strictly increasing on [a, b] if $f'(x) > 0$ for each $x \in (a, b)$.

b) f is increasing on [a, b] if $f'(x) \geq 0$ for each $x \in (a, b)$.

c) f is strictly decreasing on [a, b] if $f'(x) < 0$ for each $x \in (a, b)$.

d) f is decreasing on [a, b] if $f'(x) \leq 0$ for each $x \in (a, b)$.

e) f is constant function on [a, b] if $f'(x) = 0$ for each $x \in (a, b)$.

- **Tangents and Normals**

a) Equation of the tangent to the curve $y = f(x)$ at (x_1, y_1) is

$$y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$$

b) Equation of the normal to the curve $y = f(x)$ at (x_1, y_1) is

$$y - y_1 = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$$

- **Maxima and Minima**

a) Let f be a function and c be a point in the domain of f such that either f '(x)=0 or f '(x) does not exist are called critical points.

b) First Derivative Test: Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in I. Then

i. *f ´(x)* changes sign from positive to negative as x increases through *c*, then *c* is called the point of the local maxima.

ii. *f ´(x)* changes sign from negative to positive as *x* increases through *c*, then *c* is a point of *local minima*.

iii. $f'(x)$ does not change sign as x increases through c, then c is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion.*

c) Second Derivative Test : Let f be a function defined on an interval I and let $c \in I$. Let f be twice differentiable at c. Then

i. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of f.

ii. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value $f(c)$ is local minimum value of f.

iii. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

Very Short Answer Type Questions (1 Mark)

1. Find the angle θ, where $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

2. Find the slope of the normal to the curve $x = a\cos^3\theta$ and $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$

3. A balloon which always remains spherical has a variable radius. Find the rate at which its volume is increasing with respect to its radius when the radius is 7cm.

4. Write the interval for which the function $f(x) = \cos x, 0 \leq x \leq 2\pi$ is decreasing

5. For what values of x is the rate of increasing of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of$x$?

6. Find the point on the curve $y = x^2 - 2x + 3$ where the tangent is parallel to x-axis.

7. Write the maximum value of $f(x) = \frac{\log x}{x}$,if it exists.

8. Find the least value of $f(x) = ax + \frac{b}{x}$, where a>0, b>0 and x>0.

9. Find the interval in which the function $f(x) = x - e^x + \tan(\frac{2\pi}{7})$increases.

10. For the curve $y = (2x + 1)^3$ find the rate of change of slope of the tangent.

11. Find the value of a for which the function $f(x) = x^2 - 2ax + 6, x > 0$ is strictly increasing.

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

12. Find the co-ordinates of the point on the curie $y^2 = 3 - 4x$, where tangent is parallel to the line $2x + y - 2 = 0$

13. The sum of the two numbers is 8, what will be the maximum value of the sum of their reciprocals.

14. Find the maximum value of $f(x) = 2x^3 - 24x + 107$ in the internal [1, 3]

15. If the rate of change of Area of a circle is equal to the rate of change its diameter. Find the radius of the circle.

16. The sides of on equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when side is 10 cm.

Rate of Change (4 Mark Questions)

17. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cm^3 of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm. Why is child labour not good for society?

18. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of $\frac{3}{2}$c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

19. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of an edge of the cube.

20. A kite is moving horizontally at a height of 151.5 meters. If the speed of the kite is 10m/sec, how fast is the string being let out when the kite is 250 m away from the boy who is flying the kite ? The height of the boy is 1.5 m.

21. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 sec. and what is the average rate at which the water flows out during the first 5 seconds?

22. A man 2m tall, walk at a uniform speed of 6km/h away from a lamp post 6m high. Find the rate at which the length of his shadow increases.

23. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lower most. Its semi- vertical angle is $\tan^{-1}(0.5)$. water is poured into it at a constant rate of$5m^3/h$. Find the

rate at which the level of the water is rising at the instant, when the depth of Water in the tank is 4m.

24. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.

25. A conical vessel whose height is 10 meters and the radius of whose base is half that of the height is being filled with a liquid at a uniform rate of $1.5m^3/min.$ find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.

26. Let $x\ and\ y$ be the sides of two squares such that$y = x - x^2$. Find the rate of change of area of the second square w.r.t. the area of the first square.

27. The length of a rectangle is increasing at the rate of 3.5 cm/sec. and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.

28. If the areas of a circle increases at a uniform rate, then prove that the perimeter various inversely as the radius.

Increasing and Decreasing [4 MARK QUESTIONS]

29. Show that $f(x) = x^3 - 6x^2 + 18x + 5$ is an increasing function for all$x \in R$. Find its value when the rate of increase of f(x) is least.

[Hint: Rate of increase is least when $f'(x)$ is least.]

30. Determine whether the following function is increasing or decreasing in the given interval: $f(x) = \cos\left(2x + \frac{\pi}{4}\right), \frac{3\pi}{8} \leq x \leq \frac{5\pi}{8}$.

31. Determine for which values of x, the function y=$x^4 - \frac{4x^3}{3}$ is increasing and for which it is decreasing.

32. Find the interval of increasing and decreasing of the function $f(x) = \frac{\log x}{x}$

33. Find the interval of increasing and decreasing of the function $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$.

34. Show that $f(x) = x^2 e^{-x}, 0 \leq x \leq 2$ is increasing in the indicated interval.

35. Prove that the function $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

36. Find the intervals in which the following functionis decreasing.

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

37. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$ is strictly decreasing.

38. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$,is strictly increasing the interval $\left(0, \frac{\pi}{4}\right)$.

39. Find the interval in which the function $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is increasing or decreasing.

40. Find the interval in which the function given by

$$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$$

(1) strictly increasing

(2) strictly decreasing

Tangent and Normal [4 MARK QUESTIONS]

41. Find the equation of the tangent to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$at the point $(\sqrt{2}a, b)$.

42. Find the equation of the tangent to the curve$y = x^2 - 2x + 7$ which is

    (1) Parallel to the line $2x - y + 9 = 0$

    (2) Perpendicular to the line $5y - 15x = 13$

43. Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

44. Find a point on the parabola $f(x) = (x - 3)^2$ where the tangent is parallel to the chord joining the points (3,0) and (4,1).

45. Find the equation of the normal to the curve $y = e^{2x} + x^2$ at $x = 0$. also find the distance from origin to the line.

46. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point, where the curve intersects the axis of y.

47. At what point on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangent is parallel to

    (1) X – axis

    (2) Y – axis

48. Show that the equation of the normal at any point 'θ' on the curve$x = 3\cos\theta - cos^3\theta$, $y = 3\sin\theta - sin^3\theta$ is
    $4(y\,cos^3\theta - xsin^3\theta)$=3 sin 4 θ.

49. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

50. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 Units/sec. then how fast is the slope of the curve changing when x=3?

51. Find the condition for the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$ to interest orthogonally.

52. Show that the curves $y = a^x$ and $y = b^x$, a > b > 0 intersect at an angle of $\tan^{-1}\left(\left|\frac{\log\left|\frac{a}{b}\right|}{1+\log \quad \log b}\right|\right)$

53. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

54. Find the equation of the normal at a point on the curve$x^2 = 4y$, which passes through the point (1, 2). Also find the equation of the corresponding tangent.

55. Find the point on the curve $9y^2 = x^3$ where the normal to the curve makes equal intercepts with the axes.

56. Show that the tangents to the curve $y = 2x^3 - 3$ at the point where x = 2 and x = -2 are parallel.

APPROXIMATION [4 MARK QUESTIONS]

Use differentials to find the approximate value of (Ques.57 to 62)

57. $(66)^{1/3}$ 58. $\sqrt{401}$

59. $\sqrt{0.037}$ 60. $\sqrt{25.3}$

61. $(3.968)^{3/2}$ 62. $(26.57)^{1/3}$

63. Find the value of $log_{10}(10.1)$ given that $log_{10}e = 0.4343$.

64. If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

65. If the side of a cube be increased by 0.1%, find the corresponding increase in the volume of the cube.

66. Find the approximate value of f(2.01) where $f(x) = x^3 - 4x + 7$.

67. Find the approximate value of $\frac{1}{\sqrt{25.1}}$, using differentials.

68. The radius of a sphere shrinks from 10 cm. to 9.8 cm. Find the approximately decrease in its volume.

Maxima and Minima (4 Mark Questions)

69. Find the maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

70. Find the absolute maximum value and absolute minimum value of the following question $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$ in [-2, 2.5]

71. Find the maximum and minimum values of $f(x) = x^{50} - x^{20}$ in the interval [0, 1]

72. Find the absolute maximum and absolute minimum value of $f(x) = (x-2)\sqrt{x-1}$ in [1, 9]

73. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Maxima and Minima (4/6 Mark Question)

74. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$.

75. If the sum of length of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

76. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

77. The sum of the surface areas of cuboids with sides x, 2x and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum if x = 3 radius of the sphere. Also find the minimum value of the sum of their volumes.

78. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

79. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere.

80. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

81. Show that the volume of the greatest cylinder which can be inscribed in a cone of height H and semi-vertical angle α is $\frac{4}{27}\pi h^3 tan^2\alpha$. Also show that height of the cylinder is $\frac{h}{3}$

82. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2,1).

83. Find the shortest distance between the line $y - x = 1$ and the curve$x = y^2$.

84. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum?

85. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$.

86. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answers

1. $\frac{\pi}{3}$
2. 1
3. $196\pi \frac{cm^3}{cm}$
4. $[0,\pi]$
5. $3,\frac{1}{3}$
6. (1,2)
7. $\frac{1}{e}$
8. $2\sqrt{ab}$
9. $(-\infty,0)$
10. 0
11. $a \leq 0$
12. $\left(\frac{1}{2},1\right)$
13. $\frac{1}{2}$
14. 89
15. $\frac{1}{\pi}$ units
16. $10\sqrt{3}\ cm^2 / s$
17. $\frac{1}{\pi}\ cm / s$
18. $\frac{3}{8\pi}\ cm / min$
20. 8 m/sec.
21. 3000 L/s
22. 3 km/h
23. $\frac{35}{88}$ m/h
25. $\frac{6}{49\pi}$ m/min.
26. $1 - 3x + 2x^2$
27. 8 $\mathrm{cm}^2/\mathrm{sec}$

29. 25

30. Increasing

31. Increasing for all $x \geq 1$

    Decreasing for all $x \leq 1$

32. Increasing on $\left(o, e\right)$

    Decreasing on [e, ∞)

33. Increasing on

$$\left(O, \frac{3\pi}{4}\right) U \left(\frac{7\pi}{4}, 2\pi\right)$$

    Decreasing on $[\frac{3\pi}{4}, \frac{7\pi}{4}]$

36. $(-\infty, 1] U [2, 3]$

37. [1,∞]

39. increasing on [0, ∞)

    Decreasing (-∞, 0]

40. Strictly increasing

    [-2,1] U [3, ∞)

    Strictly decreasing

    (-∞, -2] U [1, 3]

46. $\sqrt{2}bx - ay - ab = 0$

47. (1)$y - 2x - 3 = 0$

    (2)$36y + 12x - 227 = 0$

43. (4, 4)

44. $\left(\frac{7}{2}, \frac{1}{4}\right)$

45. $2y + x - 2 = 0$, $\frac{2}{\sqrt{5}}$

47. (1) (1, 0) and (1, 4)

    (2) (3, 2) and (-1, 2)

50. decrease 72 units/sec.

51. $a^2 = b^2$

53. $2x + 3my - 3a\, m^4 - 2am^2 = 0$

54. $x + y = 3$, $y = x - 1$

55. $\left(4, \pm\frac{8}{3}\right)$

57. 4.042

58. 20.025

59. 0.1924

60. 5.03

61. 7.904

62. 2.984

63. 1.004343

64. $\pi\ cm^2$

65. 0.3%

66. 7.08

67. 0.198

68. $80\pi cm^3$

69. max. value $=\frac{3}{4}$, mim value $=\frac{1}{2}$

70. ab. Max.$=\frac{157}{8}$, ab. Min. $=\frac{-7}{4}$

71. max.value=0,

min.value $=\frac{-3}{5}\left[\frac{2}{5}\right]^{2/3}$

72. ab. Max = 14 at x = 9

ab. Min.$=\frac{-3}{4^{4/3}}$ at $x=\frac{5}{4}$

73. π

77. $18r^3+\frac{4}{3}\pi r^3$

82. (1, 2)

83. $\frac{3\sqrt{2}}{8}$

84. $\frac{144}{\pi+4}m, \frac{36\pi}{\pi+4}m$

86. 2ab sq. Units.