CMI BSc entrance exam on May 22, 2022

Part A, Correct answers

1. True
2. True
3. True
4. True
5. True
6. True
7. True
8. False
9. True
10. False
11. True
12. True
13. True
14. True
15. True
16. True
17. True
18. True
19. False
20. True
21. True
22. False
23. True
24. False
25. True
26. False
27. False
28. True
29. False
30. True
31. True
32. True
33. True
34. True
35. True
36. False
37. True
38. False
39. False
40. False

CHENNAI MATHEMATICAL INSTITUTE

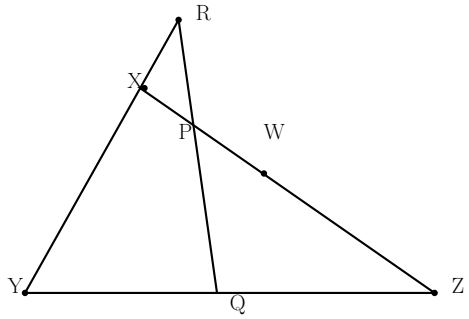
Undergraduate Programme in Mathematics and Computer Science/Physics
Solutions of the 22nd May 2022 exam

Note: The solutions below consist only of main steps and strategies and do not contain all the details expected in the exam.

B1. [**11 points**] Given $\triangle XYZ$, the following constructions are made: mark point W on segment XZ, point P on segment XW and point Q on segment YZ such that

$$\frac{WZ}{YX} = \frac{PW}{XP} = \frac{QZ}{YQ} = k.$$

See the schematic figure (not to scale). Extend segments QP and YX to meet at the point R as shown. Prove that $XR = XP$.



Solution: First a construction - mark V on XZ such that QV is parallel to YR. There are two cases here depending on whether V is between PW or WZ, however, the arguments are the same. We assume here that V is between PW. The aim is to show that $\triangle VPQ$ is isosceles and then show that it is similar to $\triangle XPR$. Use BPT to conclude that $VQ = \frac{k}{k+1}(XY)$. Using the given ratios find an expression for VZ and substitute it in $PV = PZ - VZ$ to conclude that $PV = VQ$.

One can also extend ZX to ZX' such that YX' is parallel to PQ. One can then show that $\triangle YXX'$ is isosceles and similar to $\triangle RXP$.

Another strategy is to use Menalaus theorem for $\triangle XYZ$ with segment QR as the transversal. We have:

$$\frac{XR}{RY}\frac{YQ}{QZ}\frac{ZP}{PX} = -1.$$

This leads to the following implications leading to the equality we want:

$$\begin{aligned} XR \cdot PZ &= RY \cdot PW \\ \frac{XY + XR}{XR} &= \frac{PW + WZ}{PW} \\ \frac{XY}{XR} &= \frac{WZ}{PZ} \\ \frac{PW}{XR} &= \frac{PW}{XP}. \end{aligned}$$

B2. [**11 points**] In the XY plane, draw horizontal and vertical lines through each integer on both axes so as to get a grid of small 1×1 squares whose vertices have integer coordinates.

1. Consider the line segment D joining $(0, 0)$ with (m, n). Find the number of small 1×1 squares that D cuts through, i.e., squares whose interiors D intersect. For example, the line segment joining $(0, 0)$ and $(2, 3)$ cuts through 4 small squares.

2. Now let L be an arbitrary line. Find the maximum number of small 1×1 squares in an $n \times n$ grid that L can cut through.

Solution: Assume $\gcd(m, n) = 1$. The line D has to cross $m-1$ vertical as well as horizontal lines. Moreover, D doesn't pass through any grid points. Hence, together with the starting square, we see that D cuts through $m + n - 1$ squares.
Let $\gcd(m, n) = d$. The above argument is valid from $(0, 0)$ to $(m/d, n/d)$ and so on for d many sections of D. Therefore the total number of squares D cuts is $m + n - d$.
Note that in order for L to cut through maximum number of squares it should not pass through any internal grid point. This is possible for a line joining $(0, 0)$ with (x, n) where $n - 1 < x < n$. The required answer is $2n - 1$.

B3. [**14 points**] For a positive integer n, let $f(x) := 1 + x + x^2 \cdots + x^n$. Find the number of local maxima of $f(x)$. Find the number of local minima of $f(x)$. For each maximum/ minimum $(c, f(c))$, find the integer k such that $k \leq c < k + 1$.

Solution: We have $f'(x) = 1+2x+\cdots+nx^{n-1}$. For $x \geq 0$ the derivative is strictly positive, hence $f(x)$ is strictly increasing. Therefore, we should only analyze negative values of x. Write the derivative as the following rational function

$$f'(x) = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

Note that there is no problem in the expression since we are assuming $x < 0$. Denote by $D(x)$ the denominator of the derivative.
The case when n is odd. For $x < 0$ the polynomial $D(x)$ is strictly positive. Hence there can't be any critical point.
The case when n is even. Observe that there could be only one critical point $c \in (-1, 0)$. Since $D(x) < 0$ for $x \leq -1$ and $D(0) = 1$. Moreover, $D'(x) > 0$ for $x < 0$ so $f'(x)$ is increasing on $(-\infty, 0)$ hence it vanishes exactly once. As the derivative changes sign from $-$ve to $+$ve passing through c, so there is exactly one global minimum at c (where, $-1 < c < 0$).

B4. [**14 points**] For a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, define

- A_r = the area bounded by the graph of f, X-axis, $x = 1$ and $x = r$.
- B_r = the area bounded by the graph of f, X-axis, $x = r$ and $x = r^2$.

Find all continuous functions f for which $A_r = B_r$ for every positive number r.

Solution: We are given

$$\int_1^r f(x)dx = \int_r^{r^2} f(x)dx.$$

Applying d/dr, fundamental theorem of calculus and the chain rule to above equality we get

$$f(x) = xf(x^2) \quad \forall x \in \mathbb{R}_+.$$

Letting $g(x) = xf(x)$ we see that $g(x) = g(x^2)$ for all x in the domain. Hence $g(x) = g(x^{\frac{1}{2^n}})$ for all x and positive integers n. However, as n goes to infinity $x^{\frac{1}{2^n}}$ tends to 1 we have that $g(x)$ converges to $f(1)$. Hence $xf(x) = f(1)$ for all values of $x \in \mathbb{R}_+$.

B5. [**14 points**] Two distinct real numbers r and s are said to form a good pair (r, s) if

$$r^3 + s^2 = s^3 + r^2.$$

1. Find a good pair (a, l) with the largest possible value of l. Find a good pair (s, b) with the smallest value of s. For every good pair (c, d) other than the two you found, show that there is a third real number e such that (d, e) and (c, e) are good pairs.

2. Show that there are infinitely many good pairs of rational numbers.

Solution: Consider the function $f(x) = x^3 - x^2$. Therefore (r, s) is a good pair iff $f(r) = f(s)$.
Observe that $x = 0, \frac{2}{3}$ are the only critical points of f. The local maximum occurs at $x = 0$. The line $y = 0$ intersects the graph of $f(x)$ at $(0, 0)$ and $(1, 0)$. Hence the required good pair (a, l) with the largest l value is $(0, 1)$.
Note that the local minimum occurs at $x = \frac{2}{3}$. The line $y = f(\frac{2}{3}) = \frac{-4}{27}$ intersects the graph at $(\frac{-1}{3}, \frac{-4}{27})$ and $(\frac{2}{3}, \frac{-4}{27})$. Hence the required good pair is $(\frac{-1}{3}, \frac{2}{3})$.
For $k \in (\frac{-4}{27}, 0)$ the line $y = k$ intersects the graph at 3 points. Hence the last statement of the first part follows.
For the second part we need to show that there for every rational number $q \in (\frac{-4}{27}, 0)$ the equation

$$x^3 - x^2 - q = 0$$

has infinitely many rational solutions. However, this is true because there are infinitely many rationals satisfying $c + d + e = 1, cd + de + ce = 0, cde = q$.

B6. [**14 points**] Solve the following.

1. Let p be a prime. Show that $x^2 + x - 1$ has at most two roots modulo p. Find all primes p for which there is exactly one root.

2. Find all positive integers $n \leq 121$ such that $n^2 + n - 1$ is divisible by 121.

3. What can you say about the number of roots of this equation modulo p^2.

Solution: Let a, b be two distinct roots of the equation modulo p. Therefore, p divides $a^2 + a - 1 - (b^2 + b - 1)$, which is equivalent to saying that p divides either $a - b$ or $a + b + 1$. In the former case we will have $a = b$, which is not allowed. Since both a, b are between 1 and p we have $3 \leq a + b + 1 \leq 2p - 3$ which implies $a + b + 1 = p$. Thus $b = p - a - 1$ is uniquely determined.
Suppose a is the only root. Then $p - a - 1 = a$, i.e., $p = 2a + 1$. Therefore, $2a + 1$ divides $4(a^+a - 1)$ and $(2a + 1)^2$. Subtracting we get that $2a + 1$ divides 5.
Part 2: Since 121 divides $n^+n - 1$, 11 also divides it. Note that $n^2 + n - 1$ and $n^2 + n - 12$ are congruent modulo 11. So the roots of the equation are $7, 3$ modulo 11.
Consider $n = 3 + 11k$. Then $n^2 + n - 1$ is congruent to $77k + 11$ modulo 121. Then $k = 3$ works giving us $n = 36$. Now consider $n = 7 + 11k$. In that case, $n^+n - 1$ is congruent to $165k + 55$ modulo 121. Which gives us $k = 7$ and $n = 84$.
For part (3), let a be a root modulo p. Then n is of the form $kp + a$ for some k between 0 and $p - 1$. We would like to solve for k the following equation

$$(kp + a)^2 + (kp + a) - 1$$

modulo p^2. This is equivalent to finding k such that p divides $k(2a + 1) + a^2 + a - 1$. If $2a + 1$ is not a multiple of p then $k = -(2a + 1)^{-1}(a^2 + a - 1)$. If p divides $(2a + 1)$ then it is 5 and there is no such n.