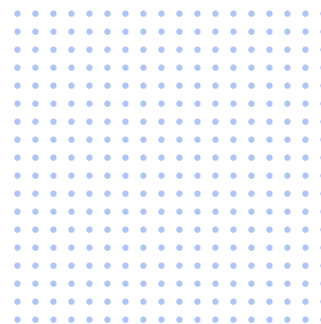




FOR ISI ADMISSION TEST EXAM PREPARATION

# ISI Admission Test 2020 Question Paper · M.Math PMB

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EXAM

ISI Admission Test

YEAR

2020

TYPE

Question Paper

SUBJECT

M.Math PMB

**Notation.** In the following,  $\mathbb{R}$  denotes the set of real numbers.

(1) (a) Let  $\{f_n\}$  be a sequence of continuous real-valued functions on  $[0, 1]$  converging uniformly on  $[0, 1]$  to a function  $f$ . Suppose for all  $n \geq 1$  there exists  $x_n \in [0, 1]$  such that  $f_n(x_n) = 0$ . Show that there exists  $x \in [0, 1]$  such that  $f(x) = 0$ .

(b) Give an example of a sequence  $\{f_n\}$  of continuous real-valued functions on  $[0, \infty)$  converging uniformly on  $[0, \infty)$  to a function  $f$ , such that for each  $n \geq 1$  there exists  $x_n \in [0, \infty)$  satisfying  $f_n(x_n) = 0$ , but  $f$  satisfies  $f(x) \neq 0$  for all  $x \in [0, \infty)$ .

(2) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Show that

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \left( 1 + \frac{1}{n} f\left(\frac{k}{n}\right) \right) = e^{\int_0^1 f(x) dx}.$$

(3) (a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a twice continuously differentiable function. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

for all  $x \in \mathbb{R}$ .

(b) Show that if  $f$  further satisfies

$$\frac{1}{2y} \int_{x-y}^{x+y} f(t) dt = f(x)$$

for all  $x \in \mathbb{R}, y > 0$ , then there exist  $a, b \in \mathbb{R}$  such that  $f(x) = ax + b$  for all  $x \in \mathbb{R}$ .

(4) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a twice continuously differentiable function. Show that if  $f$  is bounded and  $f''(x) \geq 0$  for all  $x \in \mathbb{R}$  then  $f$  must be constant.

(5) Let  $J$  be a  $2 \times 2$  real matrix such that  $J^2 = -I$ , where  $I$  is the identity matrix.

(a) Show that if  $v \in \mathbb{R}^2$  and  $v \neq 0$ , then the vectors  $v, Jv \in \mathbb{R}^2$  are linearly independent.

(b) Show that there exists an invertible  $2 \times 2$  real matrix  $U$  such that

$$UJU^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(6) Suppose  $V$  is a 3-dimensional real vector space and  $T : V \rightarrow V$  is a linear map such that  $T^3 = 0$  and  $T^2 \neq 0$ .

(a) Show that there exists a vector  $v \in V$  such that the set  $\{v, T(v), T^2(v)\}$  is a basis of  $V$ .

(b) Suppose  $S : V \rightarrow V$  is another linear map such that  $S^3 = 0$  and  $S^2 \neq 0$ . Show that there exists an invertible linear map  $U : V \rightarrow V$  such that  $S = UTU^{-1}$ .

(7) Let  $K$  be a field, and let  $R$  be the ring  $K[x]$ . Let  $I \subset R$  be the ideal generated by  $(x - 1)(x - 2)$ . Find all maximal ideals of the ring  $R/I$ .

(8) Let  $G$  be a finite group, and let  $H$  be a normal subgroup of  $G$ . Let  $P$  be a Sylow  $p$ -subgroup of  $H$ .

(a) Show that for all  $g \in G$ , there exists  $h \in H$  such that  $gPg^{-1} = hPh^{-1}$ .

(b) Let  $N = \{g \in G | gPg^{-1} = P\}$ . Let  $HN$  be the set  $HN = \{hn | h \in H, n \in N\}$ . Show that  $G = HN$ .