

Test I: UGA

30 MCQs · 120 marks

Test II: UGB

8 problems · 80 marks

Time

2 h + 2 h

Calculators

Not allowed

Part	Section	Questions	Q. No.
Test I: UGA (2 hours, 120 marks)	UGA Group A: exactly one correct option	20	1 to 20
Test I: UGA (2 hours, 120 marks)	UGA Group B: one or more correct options	10	21 to 30
Test II: UGB (2 hours, 80 marks)	UGB: descriptive problems (10 points each)	8	31 to 38

Instructions

1. This practice paper has the two tests that B Stat and B Math candidates write on the same day: Test I (UGA, forenoon, 2 hours) and Test II (UGB, afternoon, 2 hours).
2. UGA has 30 multiple-choice questions with four options each, worth 4 marks each (total 120). Group A (Q1–20): exactly ONE option is correct. Group B (Q21–30): ONE OR MORE options may be correct, and you must identify all of them.
3. ISI's 2026 prospectus and General Instructions to Candidates do not state a negative-marking rule for UGA. To score yourself, give 4 marks for a fully correct answer and 0 otherwise (in Group B, all the correct options and no wrong option), and follow the marking rule printed on the 2027 question booklet.
4. In the real test UGA answers are shaded on an OMR sheet with a black ball-point pen. Here, write your choice(s) in the box under each question.
5. UGB has 8 problems of 10 points each (total 80). Write complete solutions with proofs on separate sheets; the answer booklet is supplied in the real test. Only candidates above the UGA qualifying score have their UGB scripts graded.
6. Calculators, mobile phones, log tables and other aids are not allowed.
7. The answer key and full solutions are at the end. In the key, Group B shows the set of correct options and UGB shows the final answer where the problem has one ("—" for pure proofs).

This is an AglaSem practice paper written on the ISI Admission Test 2026 B Stat / B Math pattern (UGA: 20 single-correct + 10 multiple-correct MCQs, 120 marks; UGB: 8 descriptive problems, 80 marks; 2 hours each) and the official UGA/UGB syllabus. It is not an official Indian Statistical Institute paper, and its questions are original.

Test I: UGA (2 hours, 120 marks)

UGA Group A: exactly one correct option

1. The number of positive integers $n \leq 1000$ for which $n^2 + n + 1$ is divisible by 7 is
(A) 285 (B) 286
(C) 287 (D) 143
2. The last two digits of $3^{2^{27}}$ are
(A) 43 (B) 27
(C) 87 (D) 67
3. The value of $\lim_{n \rightarrow \infty} (n!)^{1/n} / n$ is
(A) e (B) 0
(C) 1 (D) $1/e$

4. The value of $\int_0^1 \ln(1+x)/(1+x^2) dx$ is
 (A) $(\pi/8) \ln 2$ (B) $\pi/8$
 (C) $(\pi/2) \ln 2$ (D) $(\pi/4) \ln 2$
5. The number of real solutions of the equation $x^2 = 2^x$ is
 (A) 2 (B) 1
 (C) 3 (D) 4
6. The number of ordered triples (a, b, c) of positive integers with $abc = 1000$ is
 (A) 81 (B) 64
 (C) 120 (D) 100
7. A triangle has sides 13, 14 and 15. The distance between its incentre and its circumcentre is
 (A) $1/2$ (B) $\sqrt{65}/8$
 (C) $\sqrt{65}/4$ (D) $65/64$
8. Two numbers a and b are chosen independently and uniformly at random from the interval $[0, 1]$. The probability that the equation $x^2 + ax + b = 0$ has real roots is
 (A) $1/6$ (B) $1/4$
 (C) $1/3$ (D) $1/12$
9. Let $z \neq 1$ be a complex number with $z^7 = 1$. Then $z/(1+z^2) + z^2/(1+z^4) + z^3/(1+z^6)$ equals
 (A) -2 (B) 2
 (C) 0 (D) -1
10. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients such that $p(1) = 10$, $p(2) = 20$ and $p(3) = 30$. Then $[p(12) + p(-8)]/10$ equals
 (A) 2027 (B) 1980
 (C) 1990 (D) 1984
11. Define $a_1 = 1$ and $a_{n+1} = a_n + 1/a_n$ for $n \geq 1$. Then the integer part of a_{100} is
 (A) 14 (B) 10
 (C) 15 (D) 13
12. The number of subsets of $\{1, 2, \dots, 10\}$ (including the empty set) that contain no two consecutive integers is
 (A) 89 (B) 512
 (C) 144 (D) 233
13. If a, b, c are positive real numbers with $a + b + c = 1$, then the minimum value of $1/a + 4/b + 9/c$ is
 (A) 14 (B) 36
 (C) 49 (D) 18
14. The value of $\tan^2(\pi/7) + \tan^2(2\pi/7) + \tan^2(3\pi/7)$ is
 (A) 35 (B) 21
 (C) 14 (D) 7
15. Which of the following numbers is the largest?
 (A) $4^{1/4}$ (B) $5^{1/5}$
 (C) $3^{1/3}$ (D) $2^{1/2}$
16. The locus of the midpoints of the chords of the parabola $y^2 = 4x$ that pass through its focus is
 (A) $y^2 = 2x - 1$ (B) $y^2 = 4(x - 1)$
 (C) $y^2 = x - 1$ (D) $y^2 = 2(x - 1)$

17. The area of the region bounded by the curves $y = |x^2 - 4|$ and $y = 5$ is
 (A) $40/3$ (B) $44/3$
 (C) $52/3$ (D) 16
18. The number of positive divisors of $10!$ is
 (A) 270 (B) 192
 (C) 288 (D) 240
19. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y) + 2xy$ for all real x, y , and $f(1) = 3$. Then $f(3)$ equals
 (A) 15 (B) 12
 (C) 9 (D) 18
20. The number of five-digit positive integers whose digits add up to 9 is
 (A) 330 (B) 715
 (C) 495 (D) 500

UGA Group B: one or more correct options

21. Let $p(x) = x^4 - 4x^3 + 6x^2 - 4x - 15$. Then (One or more options may be correct.)
 (A) $p(x) = 0$ has exactly two real roots.
 (B) the sum of the real roots of $p(x) = 0$ is 4.
 (C) the product of all four roots of $p(x) = 0$ is -15 .
 (D) $p(x) > 0$ for every x in the interval $(-1, 3)$.
22. For $n \geq 1$ let $a_n = (1 + 1/n)^n$ and $b_n = (1 + 1/n)^{n+1}$. Then (One or more options may be correct.)
 (A) $b_n - a_n \rightarrow 0$ as $n \rightarrow \infty$.
 (B) (a_n) is strictly increasing.
 (C) $a_n < e < b_n$ for every n .
 (D) (b_n) is strictly decreasing.
23. Let A be a 2×2 matrix with real entries such that $A^2 = I$, the identity matrix. Then (One or more options may be correct.)
 (A) $\det A = 1$ or $\det A = -1$.
 (B) $A = I$ or $A = -I$.
 (C) if the trace of A is 0, then $\det A = -1$.
 (D) the trace of A is $-2, 0$ or 2 .
24. Let n be a positive integer. Which of the following statements are always true? (One or more options may be correct.)
 (A) $n^2 + n + 41$ is prime.
 (B) $n^2 + 1$ is not divisible by 3.
 (C) $n^5 - n$ is divisible by 30.
 (D) if $2^n + 1$ is prime, then n is prime.

25. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x|x|$. Then (One or more options may be correct.)

- (A) f is convex on $\mathbb{R}$.
- (B) f is differentiable at every real number.
- (C) f' is differentiable at 0.
- (D) f is one-one.

26. In a triangle ABC the sides are $a = BC = 7$, $b = CA = 8$ and $c = AB = 5$. Then (One or more options may be correct.)

- (A) the area of the triangle is $20\sqrt{3}$.
- (B) the circumradius is $7/2$.
- (C) the inradius is 2.
- (D) $\angle A = 60^\circ$.

27. Let z be a complex number with $|z| = 1$ and $z \neq -1$. Then (One or more options may be correct.)

- (A) the real part of $(1 - z)/(1 + z)$ is 0.
- (B) $z/(1 + z)^2$ is a real number.
- (C) $|(1 - z)/(1 + z)| = 1$.
- (D) $(1 + z)/(1 + \bar{z}) = z$.

28. A fair die is rolled twice. Let E be the event that the sum of the two numbers is 7, F the event that the first roll shows 4, and G the event that the second roll shows an even number. Then (One or more options may be correct.)

- (A) E and G are independent.
- (B) E , F and G are mutually independent.
- (C) F and G are independent.
- (D) E and F are independent.

29. Let $f(x) = x e^{-x}$ for real x . Then (One or more options may be correct.)

- (A) f has a point of inflection at $x = 2$.
- (B) for every c with $0 < c < 1/e$, the equation $f(x) = c$ has exactly two real solutions.
- (C) the maximum value of f on $\mathbb{R}$ is $1/e$.
- (D) $\int_0^\infty f(x) dx = 2$.

30. Let x and y be positive real numbers with $x + y = 1$. Then (One or more options may be correct.)

- (A) $(x + 1/x)^2 + (y + 1/y)^2 \geq 25/2$.
- (B) $1/x + 1/y \geq 4$.
- (C) $x^3 + y^3 \geq 1/2$.
- (D) $xy \leq 1/4$.

UGB: descriptive problems (10 points each)

31. Find all positive integers n such that $n \cdot 2^n + 1$ is a perfect square. [10 points]

32. Let a, b, c be positive real numbers with $abc = 1$. Prove that $a^2 + b^2 + c^2 \geq a + b + c$. [10 points]

33. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$, with $f(0) = 0$ and $f(1) = 1$. Prove that there exist two distinct points a, b in $(0, 1)$ such that $f'(a) \cdot f'(b) = 1$. [10 points]

34. In a triangle ABC with $BC = a$, $CA = b$ and $AB = c$, the median from B is perpendicular to the median from C . Prove that $b^2 + c^2 = 5a^2$. Deduce that $\cos A \geq 4/5$. [10 points]

35. Find the number of subsets S of $\{1, 2, \dots, 12\}$ (including the empty set) such that no two elements of S differ by exactly 2. [10 points]

36. Let $P(x)$ be a polynomial with integer coefficients such that $P(a) = P(b) = P(c) = 2$ for three distinct integers a, b, c . Prove that $P(n) \neq 3$ for every integer n . [10 points]

37. Define $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.
(a) Prove that $a_n = 2 \cos(\pi/2^{n+1})$ for every $n \geq 1$.
(b) Find $\lim_{n \rightarrow \infty} 2^n \sqrt{2 - a_n}$. [10 points]

38. A fair coin is tossed repeatedly until two consecutive heads appear for the first time. Let N be the number of tosses.

- (a) Show that $P(N = n) = F_{n-1} / 2^n$ for $n \geq 2$, where $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$.
(b) Find the expected value of N . [10 points]

Answer Key

1	B	2	C	3	D	4	A	5	C	6	D	7	B	8	D	9	A	10	D
11	A	12	C	13	B	14	B	15	C	16	D	17	B	18	A	19	A	20	C
21	A,C	22	A,B,C,D	23	A,C,D	24	B,C	25	B,D	26	D	27	A,B,D	28	A,C,D	29	A,B,C	30	A,B,D
31	2, 3	32	—	33	—	34	—	35	441	36	—	37	$\pi/2$	38	6				

Solutions

UGA Group A: exactly one correct option

1. (B) Check residues mod 7: $n^2 + n + 1 \equiv 0$ only for $n \equiv 2$ ($4 + 2 + 1 = 7$) and $n \equiv 4$ ($16 + 4 + 1 = 21$). Numbers ≤ 1000 with $n \equiv 2$: 2, 9, ..., 996, that is 143 numbers; with $n \equiv 4$: 4, 11, ..., 998, also 143. Total 286.

2. (C) $3^{10} = 59049 \equiv 49 \pmod{100}$, so $3^{20} \equiv 49^2 = 2401 \equiv 1 \pmod{100}$. As $2027 = 20 \times 101 + 7$, $3^{2027} \equiv 3^7 = 2187 \equiv 87 \pmod{100}$.

3. (D) $\log$ of the expression $= (1/n) \sum \log(k/n)$ for $k = 1$ to n , a Riemann sum for $\int_0^1 \log x \, dx = -1$. So the limit is $e^{-1} = 1/e$.

4. (A) Put $x = \tan \theta$: $I = \int_0^{\pi/4} \ln(1 + \tan \theta) \, d\theta$. Replacing θ by $\pi/4 - \theta$ gives $1 + \tan(\pi/4 - \theta) = 2/(1 + \tan \theta)$, so $I = \int_0^{\pi/4} [\ln 2 - \ln(1 + \tan \theta)] \, d\theta = (\pi/4) \ln 2 - I$. Hence $I = (\pi/8) \ln 2$.

5. (C) $x = 2$ and $x = 4$ are solutions. For $x < 0$, $h(x) = x^2 - 2^x$ is continuous with $h(-1) = 1/2 > 0$ and $h(0) = -1 < 0$, and h is strictly decreasing there (x^2 decreases, 2^x increases), so there is exactly one negative root. For $x > 0$, $\ln x/x = \ln 2/2$ has exactly two solutions since $\ln x/x$ increases on $(0, e)$ and decreases on (e, ∞) . Total 3.

6. (D) $1000 = 2^3 \cdot 5^3$. The exponents of 2 in a, b, c are non-negative integers adding to 3: ${}^5C_2 = 10$ ways; similarly 10 ways for the exponents of 5. Total $10 \times 10 = 100$.

7. (B) $s = 21$, area $= \sqrt{(21 \cdot 8 \cdot 7 \cdot 6)} = 84$, $r = 84/21 = 4$, $R = abc/(4 \cdot \text{area}) = 2730/336 = 65/8$. Euler's formula $OI^2 = R^2 - 2Rr = 4225/64 - 65 = 65/64$, so $OI = \sqrt{65/8}$.

8. (D) Real roots $\Leftrightarrow a^2 \geq 4b \Leftrightarrow b \leq a^2/4$. The required probability is the area $\int_0^1 (a^2/4) \, da = 1/12$ (note $a^2/4 \leq 1$ throughout).

9. (A) Since $z^7 = 1$, $z^{-k} = z^{7-k}$. Dividing numerator and denominator by z^k : $z/(1 + z^2) = 1/(z^{-1} + z) = 1/(z + z^6)$, $z^2/(1 + z^4) = 1/(z^2 + z^5)$, $z^3/(1 + z^6) = 1/(z^3 + z^4)$. Let $p = z + z^6$, $q = z^2 + z^5$, $r = z^3 + z^4$. Then $p + q + r = z + z^2 + \dots + z^6 = -1$, $pq + qr + rp = 2(z + z^2 + \dots + z^6) = -2$ (multiply out and reduce powers mod 7), and $pqr = 1$. So the sum is $1/p + 1/q + 1/r = (pq + qr + rp)/(pqr) = -2$. (For $z = e^{(2\pi i/7)}$, p, q, r are $2\cos(2\pi/7)$, $2\cos(4\pi/7)$, $2\cos(6\pi/7)$, the roots of $t^3 + t^2 - 2t - 1 = 0$.)

10. (D) $q(x) = p(x) - 10x$ is monic of degree 4 with roots 1, 2, 3, so $q(x) = (x - 1)(x - 2)(x - 3)(x - r)$. Then $q(12) = 990(12 - r)$ and $q(-8) = (-9)(-10)(-11)(-8 - r) = 990(8 + r)$, so $q(12) + q(-8) = 990 \times 20 = 19800$. Hence $p(12) + p(-8) = 19800 + 120 - 80 = 19840$, and the answer is 1984.

11. (A) Squaring, $a_{n+1}^2 = a_n^2 + 2 + 1/a_n^2$, so $a_{100}^2 = 1 + 2 \times 99 + \sum_{k=1}^{99} 1/a_k^2 > 199 > 196$, giving $a_{100} > 14$. Also $a_k^2 \geq 2k - 1$, so $\sum 1/a_k^2 \leq \sum 1/(2k - 1) < 1 + \int_1^{99} dx/(2x - 1) = 1 + (\ln 197)/2 < 4$. So $a_{100}^2 < 203 < 225$ and $a_{100} < 15$. Hence $[a_{100}] = 14$ (in fact $a_{100} \approx 14.21$).

12. (C) Let s_n be the count for $\{1, \dots, n\}$. A subset either omits n (s_{n-1} ways) or contains n and omits $n - 1$ (s_{n-2} ways), so $s_n = s_{n-1} + s_{n-2}$ with $s_1 = 2$, $s_2 = 3$. This gives 5, 8, 13, 21, 34, 55, 89, 144, so $s_{10} = 144$.

13. (B) By the Cauchy-Schwarz inequality, $(a + b + c)(1/a + 4/b + 9/c) \geq (1 + 2 + 3)^2 = 36$, with equality when $a : b : c = 1 : 2 : 3$, i.e. $a = 1/6$, $b = 1/3$, $c = 1/2$. So the minimum is 36.

14. (B) $\tan 7\theta = 0$ for $\theta = k\pi/7$. Writing $\tan 7\theta$ as a ratio of polynomials in $t = \tan \theta$, the numerator is $7t - 35t^3 + 21t^5 - t^7$. Its non-zero roots are $\pm \tan(k\pi/7)$, $k = 1, 2, 3$, so $u = t^2$ satisfies $u^3 - 21u^2 + 35u - 7 = 0$ with roots $\tan^2(k\pi/7)$. Their sum is 21.

15. (C) $g(x) = x^{1/x}$ has $\log g = \ln x / x$, whose derivative $(1 - \ln x)/x^2$ is positive for $x < e$ and negative for $x > e$. So among integers the maximum is at 2 or 3: $3^{1/3}$ vs $2^{1/2}$ compare as $3^2 = 9 > 2^3 = 8$ (raise both to the 6th power). Also $4^{1/4} = 2^{1/2}$, and $5^{1/5} < 4^{1/4}$. So $3^{1/3}$ is the largest.

16. (D) Points $(t^2, 2t)$ and $(s^2, 2s)$ lie on a focal chord iff $ts = -1$. The midpoint is $h = (t^2 + s^2)/2$, $k = t + s$. Then $k^2 = t^2 + s^2 + 2ts = 2h - 2$, so the locus is $y^2 = 2(x - 1)$ (this also covers the latus rectum, whose midpoint is the focus $(1, 0)$).

17. (B) The curves meet where $x^2 - 4 = 5$, i.e. $x = \pm 3$. By symmetry, Area = $2[\int_0^2 (5 - (4 - x^2)) dx + \int_2^3 (5 - (x^2 - 4)) dx] = 2[(2 + 8/3) + (9 - 19/3)] = 2[14/3 + 8/3] = 44/3$.

18. (A) $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ (exponent of 2: $5 + 2 + 1 = 8$; of 3: $3 + 1 = 4$; of 5: 2; of 7: 1). Number of divisors = $9 \cdot 5 \cdot 3 \cdot 2 = 270$.

19. (A) $g(x) = f(x) - x^2$ satisfies $g(x + y) = g(x) + g(y)$ and is continuous, so $g(x) = cx$ for a constant c . Then $f(x) = x^2 + cx$, and $f(1) = 1 + c = 3$ gives $c = 2$. So $f(3) = 9 + 6 = 15$.

20. (C) Let the digits be $d_1 \geq 1, d_2, \dots, d_5 \geq 0$ with sum 9. Put $d_1 = e + 1: e + d_2 + \dots + d_5 = 8$ in non-negative integers (each automatically $\leq 8 \leq 9$). The number of solutions is ${}^{12}C_4 = 495$.

UGA Group B: one or more correct options

21. (A,C) Correct options: (A), (C). (A) True: $p(x) = (x - 1)^4 - 16 = ((x - 1)^2 - 4)((x - 1)^2 + 4)$, with real roots 3 and -1 and non-real roots $1 \pm 2i$. (B) False: the real roots are 3 and -1 , so their sum is 2. (C) True: for a monic quartic the product of the roots is the constant term, -15 (check: $3 \cdot (-1) \cdot (1 + 2i)(1 - 2i) = -3 \cdot 5$). (D) False: for $-1 < x < 3$, $|x - 1| < 2$, so $(x - 1)^4 < 16$ and $p(x) < 0$.

22. (A,B,C,D) Correct options: (A), (B), (C), (D). (A) True: $b_n - a_n = a_n/n < 3/n \rightarrow 0$. (B) True: by AM–GM on the $n + 1$ numbers $1 + 1/n$ (n times) and 1: their mean is $1 + 1/(n + 1)$, so $a_{n+1} = (1 + 1/(n + 1))^{n+1} > (1 + 1/n)^n = a_n$ (strict as the numbers are not all equal). (C) True: both sequences converge to e ($b_n = a_n(1 + 1/n)$); an increasing sequence stays strictly below its limit and a decreasing one strictly above it. (D) True: $1/b_n = (1 - 1/(n + 1))^{n+1}$; AM–GM on $n + 1$ copies of $1 - 1/(n + 1)$ and one 1 gives $(1 - 1/(n + 2))^{n+2} > (1 - 1/(n + 1))^{n+1}$, i.e. $1/b_{n+1} > 1/b_n$.

23. (A,C,D) Correct options: (A), (C), (D). (A) True: $(\det A)^2 = \det(A^2) = 1$. (B) False: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfies $A^2 = I$. (C) True: with $t = 0$ the relation $tA = (1 + d)I$ gives $(1 + d)I = O$, so $d = -1$. (D) True: by Cayley–Hamilton, $A^2 - tA + dI = O$ with $t = \text{tr } A$, $d = \det A$, so $tA = (1 + d)I$. If $t \neq 0$, A is a scalar matrix with $A^2 = I$, so $A = \pm I$ and $t = \pm 2$; otherwise $t = 0$.

24. (B,C) Correct options: (B), (C). (A) False: for $n = 40$ it equals $40 \cdot 41 + 41 = 41^2$. (B) True: $n^2 \equiv 0$ or $1 \pmod{3}$, so $n^2 + 1 \equiv 1$ or $2 \pmod{3}$. (C) True: $n^5 - n = (n - 1)n(n + 1)(n^2 + 1)$ is divisible by 2 and 3 (product of three consecutive integers) and by 5 (Fermat: $n^5 \equiv n \pmod{5}$). (D) False: $2^4 + 1 = 17$ is prime but 4 is not (in fact $2^n + 1$ prime forces n to be a power of 2).

25. (B,D) Correct options: (B), (D). (A) False: $f''(x) = -2 < 0$ for $x < 0$, so f is concave on $(-\infty, 0)$. (B) True: $f(x) = x^2$ for $x \geq 0$ and $-x^2$ for $x < 0$; away from 0 it is a polynomial, and at 0 the difference quotient $f(h)/h = |h| \rightarrow 0$, so $f'(0) = 0$. In fact $f'(x) = 2|x|$. (C) False: $f'(x) = 2|x|$ is not differentiable at 0. (D) True: $f'(x) = 2|x| > 0$ for $x \neq 0$, so f is strictly increasing on $\mathbb{R}$.

26. (D) Correct option: (D). (A) False: area = $(1/2)bc \sin A = (1/2)(40)(\sqrt{3}/2) = 10\sqrt{3}$. (B) False: $R = a/(2 \sin A) = 7/\sqrt{3}$. (C) False: $r = \text{area}/s = 10\sqrt{3}/10 = \sqrt{3}$. (D) True: $\cos A = (b^2 + c^2 - a^2)/(2bc) = (64 + 25 - 49)/80 = 1/2$.

27. (A,B,D) Correct options: (A), (B), (D). (A) True: with $\bar{z} = 1/z$, the conjugate of $w = (1 - z)/(1 + z)$ is $(1 - 1/z)/(1 + 1/z) = (z - 1)/(z + 1) = -w$, so w is purely imaginary (or 0). (B) True: $(1 + z)^2/z = z + 2 + 1/z = 2 + 2 \operatorname{Re} z$, which is real and positive ($z \neq -1$), so its reciprocal is real. (C) False: for $z = e^{i\theta}$, $|(1 - z)/(1 + z)| = |\tan(\theta/2)|$, which is 1 only for $\theta = \pm\pi/2$; e.g. $z = 1$ gives 0. (D) True: $1 + \bar{z} = 1 + 1/z = (z + 1)/z$, so $(1 + z)/(1 + \bar{z}) = z$.

28. (A,C,D) Correct options: (A), (C), (D). (A) True: $E \cap G = \{(5, 2), (3, 4), (1, 6)\}$ has probability $3/36 = 1/12 = (1/6)(1/2)$. (B) False: $E \cap F \cap G$ is empty (first 4 and sum 7 force second 3, which is odd), so its probability is $0 \neq 1/72$. (C) True: the two rolls are independent, so $P(F \cap G) = (1/6)(1/2)$. (D) True: $P(E) = 6/36 = 1/6$, $P(F) = 1/6$ and $P(E \cap F) = P(\{(4, 3)\}) = 1/36 = P(E)P(F)$.

29. (A,B,C) Correct options: (A), (B), (C). (A) True: $f''(x) = (x - 2)e^x$ changes sign at $x = 2$. (B) True: $f < 0$ for $x < 0$; on $(0, 1)$ f increases from 0 to $1/e$ and on $(1, \infty)$ it decreases from $1/e$ towards 0, so each such c is taken once on each piece. (C) True: $f'(x) = (1 - x)e^x$, so f increases on $(-\infty, 1)$ and decreases on $(1, \infty)$; the maximum is $f(1) = 1/e$. (D) False: integrating by parts, $\int_0^\infty x e^{-x} dx = 1$.

30. (A,B,D) Correct options: (A), (B), (D). (A) True: by the QM–AM inequality the left side is at least $(x + y + 1/x + 1/y)^2/2 \geq (1 + 4)^2/2 = 25/2$, with equality at $x = y = 1/2$. (B) True: $1/x + 1/y = (x + y)/(xy) = 1/(xy) \geq 4$. (C) False: $x^3 + y^3 = (x + y)^3 - 3xy(x + y) = 1 - 3xy$, which equals $1/4$ at $x = y = 1/2$. (D) True: AM–GM: $\sqrt{(xy)} \leq (x + y)/2 = 1/2$.

UGB: descriptive problems (10 points each)

31. (2, 3) Answer: 2, 3. The required values are $n = 2$ and $n = 3$. $n = 1$ gives 3, not a square. Let $n \geq 2$ and $n \cdot 2^n + 1 = k^2$. Then k is odd, say $k = 2t + 1$, and $k^2 - 1 = 4t(t + 1)$, so $t(t + 1) = n \cdot 2^{n-2}$. Of t and $t + 1$ one is odd and the other even; the even one carries every factor 2 of the right side, so it is a multiple of 2^{n-2} , while the odd one divides n and hence is at most n . Since the two differ by 1, $2^{n-2} \leq (\text{even one}) \leq (\text{odd one}) + 1 \leq n + 1$. But $2^{n-2} > n + 1$ for $n \geq 5$ (true at $n = 5$: $8 > 6$, and the left side doubles while the right grows by 1), so $n \leq 4$. Checking: $n = 2$ gives $9 = 3^2$, $n = 3$ gives $25 = 5^2$, $n = 4$ gives 65 (not a square). So $n = 2, 3$.

32. (guidance) Let $S = a + b + c$. By AM–GM, $S \geq 3(abc)^{1/3} = 3$. By the QM–AM (or Cauchy–Schwarz) inequality, $a^2 + b^2 + c^2 \geq S^2/3 = S \cdot (S/3) \geq S \cdot 1 = S$. Hence $a^2 + b^2 + c^2 \geq a + b + c$, with equality exactly when $a = b = c = 1$.

33. (guidance) $g(x) = f(x) + x - 1$ is continuous with $g(0) = -1 < 0$ and $g(1) = 1 > 0$, so by the intermediate value theorem there is $c \in (0, 1)$ with $f(c) = 1 - c$. By the mean value theorem on $[0, c]$ there is $a \in (0, c)$ with $f'(a) = (f(c) - f(0))/c = (1 - c)/c$, and on $[c, 1]$ there is $b \in (c, 1)$ with $f'(b) = (f(1) - f(c))/(1 - c) = c/(1 - c)$. Then $a < c < b$, so $a \neq b$, and $f'(a)f'(b) = 1$.

34. (guidance) Let G be the centroid and use vectors from G : put $u = GB$ and $v = GC$; then $GA = -(u + v)$ since $GA + GB + GC = 0$. The median from B lies along GB and the median from C along GC , so $u \cdot v = 0$. Now $a^2 = |u - v|^2 = |u|^2 + |v|^2$, $b^2 = |CA|^2 = |-u - 2v|^2 = |u|^2 + 4|v|^2$, $c^2 = |BA|^2 = |-2u - v|^2 = 4|u|^2 + |v|^2$. Adding, $b^2 + c^2 = 5(|u|^2 + |v|^2) = 5a^2$. Then $\cos A = (b^2 + c^2 - a^2)/(2bc) = 4a^2/(2bc) = (2/5)(b^2 + c^2)/(bc) \geq (2/5) \cdot 2 = 4/5$, using $b^2 + c^2 \geq 2bc$ (equality when $b = c$).

35. (441) Answer: 441. Two numbers differ by 2 only if they have the same parity, so the condition splits into two independent conditions: on the odd numbers 1, 3, 5, 7, 9, 11 and on the even numbers 2, 4, ..., 12. In each chain of six numbers, consecutive terms of the chain differ by 2, so we need subsets of a 6-term chain with no two consecutive terms. As in the no-two-consecutive count, $s_n = s_{n-1} + s_{n-2}$ with $s_1 = 2$, $s_2 = 3$, giving $s_6 = 21$. The total is $21 \times 21 = 441$ (confirmed by direct enumeration of all 4096 subsets).

36. (guidance) $P(x) - 2$ has roots a, b, c , so $P(x) - 2 = (x - a)(x - b)(x - c)Q(x)$, and Q has integer coefficients (divide successively by monic linear factors). Suppose $P(n) = 3$ for an integer n . Then $(n - a)(n - b)(n - c)Q(n) = 1$, a product of integers, so each factor is ± 1 . But $n - a, n - b, n - c$ are three distinct integers, and only two integers (1 and -1) have absolute value 1. Contradiction, so $P(n) \neq 3$.

37. ($\pi/2$) Answer: $\pi/2$. (a) Induction: $a_1 = \sqrt{2} = 2 \cos(\pi/4)$. If $a_n = 2 \cos \theta$ with $\theta = \pi/2^{n+1}$, then $a_{n+1} = \sqrt{2 + 2 \cos \theta} = \sqrt{4 \cos^2(\theta/2)} = 2 \cos(\theta/2)$ ($\cos(\theta/2) > 0$), which is $2 \cos(\pi/2^{n+2})$. (b) $2 - a_n = 2(1 - \cos \theta) = 4 \sin^2(\theta/2)$, so $\sqrt{2 - a_n} = 2 \sin(\pi/2^{n+2})$. Hence $2^n \sqrt{2 - a_n} = 2^{n+1} \sin(\pi/2^{n+2}) = (\pi/2) \cdot [\sin x / x]$ with $x = \pi/2^{n+2}$, which tends to $\pi/2$ because $\sin x / x \rightarrow 1$.

38. (6) Answer: 6. (a) Call a string of H, T good if it has no two consecutive H. The number of good strings of length m is F_{m+2} (a good string ends in T after a good string of length $m - 1$, or in TH after one of length $m - 2$, so the counts satisfy the Fibonacci recurrence, with $2 = F_3$ for $m = 1$ and $3 = F_4$ for $m = 2$; the empty string gives $1 = F_2$). $N = 2$ means HH: $1 = F_1$ sequence. For $n \geq 3$, $N = n$ means tosses $n - 1$ and n are H, toss $n - 2$ is T, and the first $n - 3$ tosses form a good string: F_{n-1} choices. Each sequence of n tosses has probability 2^{-n} , so $P(N = n) = F_{n-1}/2^n$. (b) Condition on the start: if the first toss is T (probability $1/2$) we restart after 1 toss; if it is HT (probability $1/4$) we restart after 2 tosses; HH (probability $1/4$) ends in 2 tosses. So $E = (1/2)(1 + E) + (1/4)(2 + E) + (1/4)(2)$, i.e. $E = 3/2 + (3/4)E$, giving $E = 6$. (Check: $\sum_n F_{n-1}/2^n = 6$.)